torsion coefficients. Two applications of these inequalities are made. The first is to the function $f_A: X \to R$ where $A \in R^{n+1}$, $X$ is a compact $n$-manifold imbedded in $R^{n+1}$, and $f_A(x) = |x - A|$. The second applies the analysis of this distance function to prove that a Stein manifold has no integral homology past the middle dimension. This in turn yields the Lefschetz theorem relating the cohomology of a compact complex subvariety of complex projective space with that of its intersection with a complex hypersurface.

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*Foundations of constructive analysis* by Errett Bishop. McGraw-Hill, New York, 1967. xiii+370 pp. $12.00.

> For, compared with the immense expanse of modern mathematics, what would the wretched remnants mean, the few isolated results, incomplete and unrelated, that the intuitionists have obtained... (Hilbert, 1927)[1]

> While in a few cases one has succeeded in replacing certain intuitionistically void proofs by constructive ones, for the majority this has not been achieved nor is there a prospect of achieving it... (Fraenkel & Bar-Hillel, 1958)[2]

> L'école intuitionniste, dont le souvenir n'est sans doute destiné a subsister qu'à titre de curiosité historique... (Bourbaki, 1960)[3]

> Almost every conceivable type of resistance has been offered to a straightforward realistic treatment of mathematics. . . . It is time to make the attempt. (Bishop, 1967)[4]

Bishop's attempt has succeeded. Within a constructive framework intimately related to Brouwer's intuitionism—though with important differences—he has developed a substantial portion of abstract analysis, thereby arithmetizing it; and, moreover, he has done it in such a way as to establish the general feasibility and desirability of his constructivist program. He is not joking when he suggests that classical mathematics, as presently practiced, will probably cease to exist as an independent discipline once the implications and advantages of the constructivist program are realized. After more than two

[1] *The foundations of mathematics.* All quotes from Hilbert, Kolmogorov, Skolem, and Weyl are from the translations in J. van Heijenoort's *From Frege to Gödel, a source book in mathematical logic,* 1879–1931, Harvard Univ. Press, Cambridge, Mass., 1967.

[2] *Foundations of set theory,* North-Holland, Amsterdam.

[3] *Éléments d'histoire des mathématiques,* Hermann, Paris.

[4] From the first chapter of the book under review.

years of grappling with this mathematics, comparing it with the classical system, and looking back into the historical origins of each, I fully agree with this prediction.

Bishop's program is designed to study that same underlying mathematics that all of us look at one way or another, but which most of us have seen only from within the classical system. On the one hand, this program is firmly attached to the positive integers in a way that gives a natural concrete meaning to its results; on the other hand, it is expressed in terms of a sharpened, but completely general, version of real Cantorian set theory, and this makes it very attractive and familiar to the classical mathematician.

Still, much of this mathematics appears to contradict some of our basic assumptions about the nature of mathematics. So what shall we make of it? Before I try to answer this question—at some length—I will first discharge my basic service as reviewer by describing the content of Bishop's book, as if the reader were already somewhat familiar with the constructive point of view. I hope by doing this to convey some of the flavor of the mathematics and lend substance to the more general discussion that will follow.

**The review.** This book is a course in abstract analysis, starting from first principles, "the primitive concept of the unit, the concept of adjoining a unit, and the process of mathematical induction," and reaching in various directions as far as the Chacon-Ornstein ergodic theorem, Fourier analysis on groups, and Gelfand's theory of the spectrum. Besides the first chapter, which explains the constructivist program and presents Brouwer's famous critique of the deficiencies in meaning of classical mathematics, the most important chapters are those on *metric spaces* (completeness, compactness, locally compact spaces), *measure* (measures as functionals, measure of sets, measures on $R$, approximation by compact sets), and *normed linear spaces* ($L_p$ spaces, extension of linear functionals, Hilbert space and the spectral theorem, locally convex spaces, extreme points).

There are three supporting chapters: a thorough treatment of *calculus and the real number system*, the basics of *complex analysis* (Cauchy's formula, estimates of size and zeroes of analytic functions, Riemann mapping), and a short excursion into *set theory*, mainly to introduce Borel sets and to formulate an affirmative notion of complementation needed for a good measure theory.

Real numbers are defined in terms of successive rational approximations $x_n$ to within $1/n$ (i.e. a real number is a sequence $x \equiv (x_n)$ of rationals such that $|x_n - x_m| \leqq 1/n + 1/m$), and two such are defined to be equal if their termwise differences are at most $2/n$. A real number is *positive* if, for some $n$, $x_n > 1/n$ and *nonnegative* if, for all $n$, $x_n \geqq -1/n$. One could also make a definition of real numbers in terms of Dedekind cuts or arbitrary Cauchy sequences of rationals. Classically this is more elegant, but constructively it is less elegant. (Constructively, a Cauchy sequence of rationals is a sequence $(x_n)$ of rationals *and* a sequence $(N_k)$ of integers such that $|x_n - x_m| \leqq 1/k$ for $n$, $m \geqq N_k$.)

Of the remaining chapters, one is a fairly standard treatment of *Lebesgue integration* (abstract measure spaces, convergence theorems), and this provides a natural setting for the study of the classical discontinuous functions. Three chapters are somewhat more specialized: *locally compact Abelian groups*—this is quite elegant (Haar integral, convolution, Fourier inversion, and Pontryagin duality), *commutative Banach algebras* (according to the author, "the only instance in this book of a classical theory whose constructive version seems forced and unnatural"), and *limit operations in measure theory* (containing a new general ergodic theorem, in terms of upcrossing inequalities, which yields constructive versions of Doob's martingale theorem and Lebesgue's theorem that a function of bounded variation has a derivative almost everywhere).

Finally, there are two appendices. One is a philosophical addendum on the role of contradiction and on the computational meaning of the mathematics of the book. The latter subject is pursued much further in Bishop's recent essay *Mathematics as a numerical language.*[5]

The other appendix is a brief defense of the author's nearly exclusive restriction to metric spaces and his free use of separability hypotheses. His conclusion is that, at least for the parts of analysis treated in his book, this is the right setting. There are no constructively defined metric spaces which are known to be nonseparable and, indeed, all the spaces which arise naturally are separable. (The metric induced classically by the supnorm on $l_\infty$ is not constructively everywhere well defined!) Although the concept of a uniform space, defined by a set of pseudometrics, would appear promising, Bishop finds that even those uniform structures naturally associated with important locally convex spaces are not too significant constructively. For the dual of a separable Banach space a "double-norm" works better; for spaces of distributions there are other considerations.

[5] Proceedings of a Symposium on Intuitionism and Proof Theory, North-Holland, Amsterdam (to appear).

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Now I would like to use Bishop's work as a base for comparing classical and constructive mathematics. These systems can be compared in at least two different ways, depending on whether one's chief concern is to constructivize classical mathematics or, rather, to develop, on its own terms, a completely constructive mathematics. Bishop's program, like Brouwer's (and before that, Kronecker's) is the latter. But his approach, as reflected in his book, has been to base his work *initially* on the mathematics that now exists: to analyze classical mathematics from the constructive standpoint and then to use the results as a guide for further development. It is precisely this attitude that has enabled Bishop to demonstrate to the classical mathematician what the intuitionists (for whatever reasons) did not: that to replace the classical system by the constructive one does not in any way mutilate the great classical theories of mathematics. Not at all. If anything, it strengthens them, and shows them, in a truer light, to be far grander than we had known. Read Weil's description of Kronecker's constructivist program in his essay, *Number-theory and algebraic geometry*.[6]

> He was, in fact, attempting to describe and to initiate a new branch of mathematics, which would contain both number-theory and algebraic geometry as special cases. This grandiose conception has been allowed to fade out of sight. . .

At any rate, for the purposes of this discussion it will be useful to compare classical and constructive mathematics in both of the ways mentioned above.

**Constructivizing classical mathematics.** Concerning the first way, Bishop writes in his first chapter, *A constructivist manifesto*,

> The extent to which good constructive substitutes exist for the theorems of classical mathematics can be regarded as a demonstration that classical mathematics has a substantial underpinning of constructive truth.

This directly contradicts certain beliefs of Hilbert (and others) which, in reaction to Brouwer's critique, led to the modern approach of systematically suppressing constructive considerations.

> The theorems of the theory of functions, such as the theory of conformal mapping and the fundamental theorems in the theory of partial differential equations or of Fourier series—to single out only a few examples from our science—are merely ideal proposi-

[6] Proceedings of the 1950 International Congress of Mathematicians.

> tions in my sense and require the logical $\epsilon$-axiom[7] for their development. (Hilbert, 1927)[1]

> The classical foundations of *calculus*, all the more the modern theory of *real functions*, including the Lebesgue integral, clearly become meaningless in this light. (Fraenkel & Bar-Hillel, 1958)[2]

Fortunately these pessimistic views have now been shown to be totally wrong.

From the standpoint of constructivizing classical mathematics, one naturally seeks the strongest and most useful versions of classical theorems, definitions, and theories. In this spirit, purely negativistic concepts should be kept to an absolute minimum, if not completely eliminated.[8] For instance, to show that two constructively defined real numbers are not equal one must produce a positive integer $k$ such that the distance between them is at least $1/k$. In the same way, to prove that a set is nonvoid one must construct an element of it.

There is no fixed general method for determining *good* constructive substitutes for classical theorems, but the following examples give an indication of what is involved.

(i) Consider first the basic result in the theory of commutative Banach algebras that, in an algebra A with unit, any finite number of elements whose Gelfand transforms have no common zero on the spectrum generate the unit ideal. We seek a constructive interpretation of the hypotheses with the widest useful application and the strongest conclusion. We should not choose a version which assumes the compactness of the spectrum for, constructively, this is not always the case.

Bishop's method is to first construct a certain sequence $S_m$ of compact subsets of the dual space $A^*$ whose intersection is the spectrum. He then expresses the hypotheses of the theorem by considering elements $a_1, \cdots, a_n$ in $A$, and positive integers $k$ and $m$, such that, on $S_m$, $|a_1| + \cdots + |a_n| \geqq 1/k$. (This is a constructively usable form of the no common zero condition. One can always check that either it holds or else one can construct a point $x$ in $S_m$ for which $|a_1(x)| + \cdots + |a_n(x)| < 2/k$.) Under these conditions, Bishop shows how to construct $b_1, \cdots, b_n$ in $A$ such that $1 = a_1b_1 + \cdots + a_nb_n$. Also,

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[7] Hilbert's device for defining objects whose existence is deduced only by means of excluded middle.

[8] Mainly by finding stronger affirmative versions. However, in Brouwer's intuitionistic system pure negation plays a larger role.

bounds for the norms of $b_1, \cdots, b_n$ can be explicitly determined from the initial data.

In general, the explicit way constructed objects can be shown to depend—or not depend—on various portions of the initial data may constitute, even from a completely classical standpoint, a useful strengthening of the theorem which asserts the existence of these objects. An outstanding case of this is the recent work of A. Baker[9] in the theory of algebraic numbers.

(ii) When a classical assertion of convergence is not constructively valid, one may be led to study more carefully the behaviour of the approximating sequence. (Just as we do classically with Fourier series.) This can yield results which are classically substantially stronger than mere convergence. For instance, in studying Birkhoff's ergodic theorem, Bishop found that although the sequence of averages need not converge anywhere constructively, it does however obey certain upcrossing inequalities (a notion introduced by Doob for studying martingales) and, classically, this is more than convergence a.e.

(iii) The classical assertion that a uniformly continuous function on $[0, 1]$ attains its maximum at some point is not valid constructively. But there is a useful constructive substitute, for the classical proof does construct points $x_k$ where the value of the function is within $1/k$ of being an upper bound. Therefore, the supnorm is always constructively well-defined. Moreover, if it is a matter of actually determining local maxima as critical points of a smooth mapping, this too can be achieved constructively under suitable transversality assumptions.

(iv) (Noted in passing.) Of three classically equivalent formulations of the compactness of a metric space we find that, constructively, one does not apply to $[0, 1]$ (Bolzano-Weierstrass), another is apparently unprovable for $[0, 1]$ (Heine-Borel-Lebesgue),[10] and the third, as was shown by Brouwer, is excellent (complete and totally-bounded.)

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[9] *Linear forms in the logarithms of algebraic numbers.* I, II, III, IV, Mathematika **13** (1966), 204–216; ibid. **14** (1967), 102–107; ibid. **14** (1967), 220–228; **15** (1968), 204–216.

[10] Nevertheless there are good reasons why, for each specific open cover of $[0, 1]$, we should expect to quite easily determine a finite subcover. In Brouwer's intuitionism this can be made into a theorem about the processes he is describing. At this level Bishop is describing something different, and in his program the above observation has a definitely extramathematical status. Still, it finds expression within the system as a constructively valid theorem about a formalization of a portion of constructive mathematics.

(v) Classically, every set has an underlying *discrete* structure, but constructively the situation is very different. Every set is still the union of its singleton subsets, but this union is not, in general, constructively disjoint. For the rationals it is and for the reals it is not. In fact, constructively, there is no way of decomposing the real line into any disjoint union of two or more nonvoid subsets. Nevertheless, most of the interesting classical decompositions of the line (e.g. rational-irrational or positive-negative-zero) are still available constructively. However, the union of the subsets is no longer the whole line but only a certain dense subset: generally of measure one and frequently the complement of a countable or even finite set.

(vi) In the theory of normed linear spaces, giving a bounded linear functional constructively does not necessarily entail having any method for computing its norm. Nevertheless, for separable spaces, the constructively normable linear functionals are dense with respect to a natural metric on the dual space (agreeing with the weak topology on bounded subsets). Also, the strong unit ball (i.e. linear functionals bounded by 1 on the unit sphere) is constructively compact in this metric. These results suffice for most applications. For instance, the duality between $L_p$ and $L_q$ carries through in a strong form, with the modification that $L_q$ corresponds to the set of normable elements of the dual of $L_p$. One can also show that every linear functional on the dual of a separable Banach space which is continuous on the strong unit ball (in the "weak" metric) corresponds, constructively, to evaluation at a point of the space.

All these results rely on a good constructive substitute for the Hahn-Banach theorem. Besides requiring separability, one must include in the hypotheses the purely constructive restriction that the null space of the linear functional defined on the subspace be *located* in the big space (i.e. that the distance to it be constructively everywhere well defined). Under these conditions, for any positive integer $k$, one can construct a normable linear extension, with an increase in norm of at most $1/k$. The located subspaces most readily at hand are the finite-dimensional ones and, thus, many applications of Hahn-Banach are combined with an approximation of a subspace by finite-dimensional ones. A finite-dimensional space must be given with a definite basis. It is not good enough to construct finitely many elements which span.

We could go on and give more examples illustrating other kinds of phenomena that have to be taken into account when constructivizing portions of classical mathematics. In Bishop's book they fit together into a completely coherent theory. This theory now provides an excellent basis for investigating constructively other fields of analysis, such as distribution theory, partial differential equations, dynamical systems, and differential topology. Even on the level of the book there remain many interesting areas of investigation.

**The strictly constructive standpoint.** Now we will examine the classical and constructive systems from a strictly constructive point of view. Here we may ask how accurately the classical system expresses the content and meaning of constructive mathematics; also, whether it is necessary or particularly useful for discovering constructive theories, or for verifying constructive assertions.

Of course, Bishop's work demonstrates that classical mathematics is usable, at least as an initial guide, in the development of constructive mathematics. But this same work, and, above all, Brouwer's, show also that its guidance is too often unreliable, distorting, and misleading. Indeed, this should come as no surprise if we recall that Hilbert's formalization of mathematics was a deliberate attempt to "save" classical mathematics from Brouwer's critique by what Weyl called *"a radical reinterpretation of its meaning."* Therefore, even though classical mathematics does yield constructive results, and even though we may sometimes uncover patterns in its "reinterpretation" of the meaning of mathematics, from the constructive standpoint we are eventually reduced to asking only (i) is the classical system really necessary for getting constructive results and (ii) in what limited areas is it definitely reliable?

Concerning (i), Bishop writes, in his appendix on *aspects of constructive truth*,

> Hilbert's implied belief that there are a significant number of interesting theorems whose statements (standing alone) are constructive but whose proofs are not constructive (or cannot easily be made constructive) has not been justified. In fact we do not know of even one such theorem.[11]

For some, perhaps most, mathematicians, such considerations are irrelevant. Mathematics *is* classical mathematics. In that case, as Bishop writes,

> Mathematics becomes the game of sets, which is a fine game as far as it goes, with rules that are admirably precise. The game

[11] Looking at such proofs constructively tends to reveal more of the underlying structural considerations that account for the correctness of the theorem. For instance, such a theorem proved classically by appealing to the Bolzano-Weierstrass theorem may sometimes be viewed more accurately as a consequence of the pigeonhole principle.

> becomes its own justification, and the fact that it represents a highly idealized version of mathematical existence is universally ignored.

Nevertheless, as everybody knows, the modern foundations were fit underneath an already existing mathematics (which continues to go its own way) and the formal structure was superimposed. Given the current admittedly unsatisfactory state of the classical foundations, we might find it worthwhile to remind ourselves just what the formalization of mathematics was designed to accomplish and to ask how well it has succeeded. Does a formal model of axiomatic set theory really represent mathematics better than real set theory? Actually, from the constructive standpoint, we should rather first ask whether axiomatic set theory is preferable to a mathematics built directly on the integers. Also, we ought to remind ourselves why, after Brouwer's critique, the principle of excluded middle was still included in the formalization of mathematics.

Skolem, in *Some remarks on axiomatized set theory*[1] (1922), presented his famous result about countable models for set theory as a demonstration of the inadequacy of a formalistic or axiomatic approach to foundations.[12] In that same paper he wrote,

> Set-theoreticians are usually of the opinion that the notion of integer should be defined and that the principle of mathematical induction should be proved. But it is clear that we cannot define or prove ad infinitum; sooner or later we come to something that is not further definable or provable. Our only concern, then, should be that the initial foundations be something immediately clear, natural, and not open to question. This condition is satisfied by the the notion of integer and by inductive inferences, but it is decidedly not satisfied by set-theoretic axioms of the type of Zermelo's or anything else of that kind; if we were to accept the reduction of the former notions to the latter, the set-theoretic notions would have to be simpler than mathematical induction, and reasoning with them less open to question, but this runs entirely counter to the actual state of affairs.

This is precisely the constructive attitude. Offering a definition of his program, Bishop writes in *Mathematics as a numerical language*,[5]

> Thus by "constructive" I shall mean a mathematics that describes or predicts the results of certain finitely performable, albeit hypothetical, computations within the set of integers.

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[12] Because of "the fact that in every thoroughgoing axiomatization set-theoretic notions are unavoidably relative."

Hence, the constructivist program involves, first of all, a complete *arithmetization* of mathematics. Here it is following the path laid down by Descartes, Weierstrass, and others, but in the strict form first initiated by Kronecker in algebra. It is really the extension of the Kroneckerian program to all of mathematics. Not so strangely, Hilbert's formalist program which, *within* mathematics, is the antithesis of constructivism, espouses a similar sounding goal. But in neither case is this an expression of any love for arithmetic over, say, geometry. Rather, it stems from the recognition that mathematics has concrete meaning independent of logical considerations and that, moreover, for finite beings, there is an obviously intimate relation between meaningful assertions and finitely[13] verifiable ones. We get to Bishop's definition of "constructive mathematics" by the extramathematical observation that finitely verifiable assertions always reduce to the prediction of finitely performable operations among the integers.[14]

**The constructive development of mathematics.** In discussing *the descriptive basis of mathematics*, in his *constructivist manifesto*, Bishop sketches for the reader the way constructive mathematics unfolds from the integers, eventually encompassing the most general concepts of mathematics, yet always grounded in terms of descriptions of abstract finitely performable operations.

> We feel about number the way Kant felt about space. . . . Almost equal in importance . . . are the constructions by which we ascend from number to the higher levels of mathematical existence. . . . The relations which form the point of departure are the order and arithmetical relations of the positive integers. From these we construct various rules for pairing integers with one another, for separating out certain integers from the rest, and for associating integers to one another. Rules of this sort give rise to the notions of sets and functions.
>
> A set is not an entity which has an ideal existence. . . . To define a set we prescribe, at least implicitly, what we (the constructing intelligence) must do in order to construct an element of the set, and what we must do in order to show that two elements of the set are equal . . . to define a function from a set $A$ to a set $B$, we prescribe a finite routine which leads from an element of $A$ to an element of $B$, and show that equal elements of $A$ give rise to equal elements of $B$.
>
> Building on the positive integers, weaving a web of ever more

[13] The completely redundant "finitely" is just for emphasis.

[14] However, Bishop adds, "there is no reason mathematics should not concern itself with finitely performable abstract operations of other kinds, in the event that such are ever discovered; our insistence on the primacy of the integers is not absolute."[5]

> sets and more functions, we get the basic structures of mathematics: the rational number system, the real number system, . . . , the algebraic number fields, Hilbert space, . . . , and so forth. . . . Everything attaches itself to number, and every mathematical statement ultimately expresses the fact that if we perform certain computations within the set of positive integers, we shall get certain results.
>
> Mathematics takes another leap, from the entity which is constructed in fact to the entity whose construction is hypothetical. To some extent hypothetical entities are present from the start: whenever we assert that every positive integer has a certain property, in essense, we are considering a positive integer whose construction is hypothetical. But now we become bolder and consider a hypothetical set, endowed with hypothetical operations subject to certain axioms. In this way we introduce such structures as topological spaces, groups, and manifolds. The motivation for doing this comes from the study of concretely constructed examples, and the justification comes from the possibility of applying the theory of the hypothetical structure to the study of more than one specific example . . . even the most abstract mathematical statement has a computational basis.

Thus we find in the constructive framework all the basic, however general, objects of study of classical mathematics. But notice that in constructive mathematics one works, quite securely, outside any formal system, with a very general concept of sets. There is no worry about paradoxes. As Bishop puts it, "For the constructivist, consistency is not a hobgoblin. It has no independent value; it is merely a consequence of correct thought." Indeed, we find that the so-called paradoxes of Cantor's theory reside, not in the concept of a set, but rather in the truly vague notions of "ideal" existence and identity which are classically attributed to the members of any set.

Constructive mathematics is completely general in its scope, and yet it is commonly claimed that the opposite is true.[15] Usually this is caused by confusing the matter of defining a set with the problem of constructing elements of it. A typical instance of this reads, "the set consisting of 5 if Fermat's Last Theorem is true and 7 if it is false is not well defined, according to Brouwer." Not so. True, as it stands,

[15] This is the basis for most write-offs of constructivism, and some of Brouwer's earlier remarks may have unfortunately contributed to this belief. For instance, in *Intuitionism and formalism*, this Bulletin, 1913, he wrote, "the formalist introduces various concepts, entirely meaningless to the intuitionist, such as for instance '*the set whose elements are the points of space*,' '*the set whose elements are the continuous functions of a variable*' . . . and so forth."

this does not define an integer. But it does define a subset[16] of $\{5, 7\}$ containing at most one integer.

Of course, classically, we are in the same kind of situation when we consider the set of solutions $y = y(x)$ of a specific functional equation $f(x, y) = 0$ without having any particular solution at hand, and even without knowing if there are any.[17] Likewise, to define a set constructively means only to state precisely *what* must be accomplished in order to construct an element, and what else must be accomplished in order that two given elements be equal. We are not required, in defining a set, to have any way of constructing elements or any way of deciding if two elements are equal. By a *way* we mean here a finitely performable procedure. Sometimes they are at hand; but in other cases finding them will constitute a major mathematical problem.

By contrast with the ease in defining sets constructively, it is hard to construct functions. The definition of a function from a set $A$ to a set $B$ must provide an explicit way of converting the construction of any element of $A$ into the construction of a definite element of $B$. (This much describes the concept of an *operation*. To define a function we must also provide a proof that equal members of $A$ are converted into equal members of $B$.) We can say more. To be complete the definition of the function must also come with a verification that it is finitely performable. This verification may make use of other functions that have already been shown to be well defined, starting off, when all is spelled out, with the finiteness of the specific integers.

As an illustration, let us consider what must be done to define constructively a real-valued function on the closed interval, $[0, 1]$, in terms of whatever must be done to define a function from the positive integers to the integers, i.e. in terms of sequences of integers. For the sake of the present discussion, let us try to be very explicit.

It is no restriction to consider only real numbers of the form $x \equiv (x(n)/2^n)$, where $(x(n))$ is a constructively defined sequence of *integers* and, for each $n$, $x(n+1)$ differs from $2x(n)$ by at most 1. The relation of equality $x = y$ between reals is then expressed by the condition that, for all $n$, $|x(n) - y(n)| \leqq 2$. The requirement that $x$ belong to $[0, 1]$ means that, for all $n$, $-1 \leqq x(n) \leqq 2^n + 1$. Therefore, we see that the definition of any function $f: [0, 1] \rightarrow R$ must supply, *firstly*, a

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[16] Or, in Brouwer's later terminology, a "subspecies."

[17] Compare the discussion in Poincaré's essays, *Mathematics and logic* and *The logic of infinity*, (1905–06), in his *Dernières Pensées*, Dover translation, 1963.

finitely performable method for converting each definition of a sequence of integers $(x(n))$ that satisfies (a) for each $n$, $|x(n+1)-2x(n)| \leqq 1$, and (b) $-1 \leqq x(n) \leqq 2^n+1$, into the definition of another specific sequence of integers $(u_x(n))$ that satisfies $|u_x(n+1)-2u_x(n)| \leqq 1$, and, *secondly*, a proof that if, for all $n$, $|x(n)-y(n)| \leqq 2$ then, for all $n$, $|u_x(n)-u_y(n)| \leqq 2$.

To go further and define a *uniformly continuous* function we would have to provide a modulus of continuity. This amounts to defining a sequence of positive integers $(k(n))$ and giving a proof that, for all $n$, if $|x(k(n))-y(k(n))| \leqq 2$ then $|u_x(n)-u_y(n)| \leqq 2$. We note in passing that for each specific $n$ this will be a direct check.

I have taken such exceptional pains to spell everything out—though I could have gone even further—to counter the common belief that constructive mathematics must be vague and imprecise, because it—in particular, the concept of a construction—is not formalized in any way. This belief is shared by the Russian school of constructivists, and others, who feel it is necessary to base the concept of a construction on that of a recursive function. (One effect of doing this is a radical change in the way the mathematics looks.) Šanin, in *On the constructive interpretation of mathematical judgments*[18] writes that constructive mathematics "began to be developed successfully only in the middle of the 1930's after the precise mathematical concept of *arithmetic algorithm* (*computable arithmetic function*) had been worked out. Only the introduction into mathematics of the precise notion of arithmetic algorithm created a satisfactory basis for the treatment of the constructive interpretation of mathematical propositions and fundamental notions of constructive mathematical analysis." I believe that Bishop's work effectively refutes the underlying assumption here. The undefined concept of a construction actually admits a usage no less precise and clear than the undefined concept of an integer. This remarkable fact can only be obscured by bringing in recursive functions at this level.

**Extramathematical observations.** In light of the above discussion, it should not be at all surprising that Brouwer, after carefully considering how the definition of a function $f$: $[0, 1] \rightarrow R$ could possibly be given, became convinced that one would always be able to extract from it a modulus of continuity. Indeed, the status of this observation is rather like that of Church's thesis. On the one hand,

[18] In Russian (1958). Translated in Amer. Math. Soc. Transl., (2), **23** (1963), 109–189.

it is based on our possibly too limited present understanding of the way a precise definition can be given.[19] On the other hand, it sure seems like a safe bet. But, for Bishop, such extramathematical observations have no proper role *within* the actual development of constructive mathematics.[20] He writes of developing his program with an "absolute minimum of philosophical prejudice concerning the nature of constructive mathematics" and, whether or not this is really so, it expresses a pragmatic attitude which helped him to avoid a number of traps and is no doubt one of the keys to his success.

Yet this is not the whole story. Extramathematical observations can and do have great value as, more or less tentative, guiding principles in the development of constructive mathematics. Some of the most important ones, based on "counterexamples in the style of Brouwer," reveal the phenomena of nonconstructivity, and advise us not to try doing certain things constructively, e.g. defining a discontinuous function on the line. Others suggest that if we can prove constructively a certain type of result then we can in fact prove it in a strong form, e.g. that if we can prove $(r=0)\rightarrow(0=1)$, where $r$ is a constructively defined real number, then we can actually construct a positive integer $k$ and prove that $|r|>1/k$. Such observations, based on an examination of the mathematics that has already been done, are of obvious importance.

In this spirit, Bishop reviewed his own book in *Mathematics as a numerical language*[5] and, guided by an important work of Gödel,[21] made several general observations about the form of mathematical statements and about the meaning of implications, $P\rightarrow Q$. In connection with this, he also undertook to reinterpret the mathematics of his book, in a way consonant with its meaning, within a certain formal system of Gödel that is designed to accomodate constructive arithmetic and for which some of these observations reappear as constructively valid metatheorems. Moreover, considerations of this sort led to an actual improvement of some of the mathematics of the book, especially in the measure theory and the treatment of Banach algebras.

(This is as good a place as any to say—if it needs saying—that defining formal systems, constructively, and proving theorems about

[19] However, in Brouwer's scheme of things, this is already taken into account.

[20] Though such a division may ultimately be hard to maintain, it serves to give a definite shape and direction to Bishop's program.

[21] *Über eine bisher noch nich benützte Erweiterung des finiten Standpunktes*, Dialectica **12** (1958), 280–287.

them, constructively, is a part of constructive mathematics. This is so regardless of the constructive content, or lack of it, in the ideas which the system is designed to formalize.)

I am now going to describe a portion of Bishop's "review." It affords the classical mathematician a very nice view of an important part of the general structure of constructive mathematics, independent of any classical considerations.

Firstly, growing out of the "verifiability criterion for meaning," and the aim of developing a predictive mathematics, there is the concept of a *complete* mathematical statement. Such a statement asserts $\forall x \in S$, $A(x)$, where $S$ is a constructively defined set and each $A(x)$ is finitely verifiable. Included in it are the definition of $S$ and a verification that each $A(x)$ is verifiable. Of course, the definition of $S$ and the verification of verifiability may, in turn, rely on other definitions, verifications, and complete statements that have already been made—starting from the integers and inductive inferences. All of these form part of the complete statement.

One criterion for proving an existential assertion constructively, and a good one to bear in mind when trying to learn what is and what is not constructive, is that the proof, when made completely explicit, must supply a completion, in the sense above. For instance, the assertion that there exist infinitely many primes is not, as it stands, complete; but Euclid's proof provides a completion, asserting, and demonstrating, that the $(n+1)$st prime is no greater than one plus the product of the first $n$ primes.

That each $A(x)$ be finitely verifiable means precisely that the "truth value"—0 if $A(x)$ is correct, 1 if it is not—is a constructively defined integer-valued function on $S$. Thus Bishop writes,[5]

> A *complete* mathematical statement—that is, a theorem cojoined with its proof and with all theorems, proofs, and definitions on which it depends, either directly or indirectly—asserts that a given constructively defined function $f$, from a constructively defined set $S$ to the integers, vanishes identically.

However, a mathematical statement may assert $\forall x \in S$, $A(x)$, as above, without being proved. For instance, the Fermat conjecture, the Riemann hypothesis, and a good many other well-known conjectures can all, without much difficulty, be restated in the form $\forall n$, $T(n)$, where $n$ ranges over the positive integers and each $T(n)$ is finitely verifiable. (For the Fermat conjecture we can let $T(n)$ be the assertion that for all nonzero integers $a$, $b$, $c$, $d$ such that $2 < d < n$ and

$|a|+|b|+|c|<n$, we have $a^d+b^d\neq c^d$.) Similarly, any assertion that a specific real number $x\equiv(x_n)$ is nonnegative has the same form, $\forall n,\ x_n\geqq -1/n$. We might call such assertions "predictions" and reserve the adjective "complete" for ones which are accompanied by a proof. For our aim is not just to make predictions, but to make correct ones.

But the statements of mathematics are not all predictive. Not only are conjectural assertions of existence, by their nature, incomplete and nonpredictive. Even existential statements which have been proved constructively are not usually *replaced* by the completed version which the proof must provide. For good reason, because an incomplete existential theorem has a different intent from any of its completions. Namely, *that there be, at hand, some completion of it*, i.e. that there be available some specific construction of the kind of object that is asserted to exist, without bringing in the special properties of that construction. However, if no completion of a theorem appears to have any particular interest, this might suggest reformulating the original assertion another way.

Going further, Bishop defines an *incomplete* mathematical statement to be any one which asserts $\exists y\in T,\ \forall x\in S,\ A(x,y)$, where $S$ and $T$ are constructively defined sets and each $A(x,y)$ is finitely verifiable. To complete such a statement we must obviously construct some $y\in T$ and verify that $\forall x\in S,\ A(x,y)$. (For instance, any assertion that a specific real number $x\equiv(x_n)$ is positive has the incomplete form, $\exists k, x_k>1/k$. To complete it we must define a particular $k_0$ and verify that $x_{k_0}>1/k_0$.)

Bishop suggests that any assertion of constructive mathematics can be restated in this incomplete form, and this immediately raises the basic question of how we should so interpret an implication, $P\rightarrow Q$, between incomplete statements. The correct approach appears to be one that is based on an interpretation invented by Gödel,[21] which Bishop calls *numerical*, or *Gödel, implication*. He finds that it is not only harmonious with the mathematics of his book, but in fact improves it.

Gödel's interpretation is designed to be applied when each of the variables in the incomplete statements $P$ and $Q$ ranges over one of the following *special sets*: $\{1,\cdots,n\}$ is a special set. The set of positive integers is a special set. The product of two special sets is a special set. Finally, the set of all constructively defined functions from one special set to another is again a special set.

That $P$ and $Q$ can always be brought into such a form depends on certain extramathematical considerations about the way the general set of constructive mathematics is built up from these special ones.[22] At any rate, once this is arranged, if $P$ is $\exists y \forall x, A(x, y)$ and $Q$ is $\exists v \forall u, B(u, v)$, then the Gödel interpretation of $P \rightarrow Q$ is $\exists(\bar{v}, \bar{x}) \forall(y, u)$, $(A(\bar{x}(y, u), y) \rightarrow B(u, \bar{v}(y)))$, where $\bar{v}$ and $\bar{x}$ are to be constructively defined functions, with appropriate domain and range, and the parenthetical implications are to be verified in the classical manner (by checking each end of the arrow and comparing truth values). This is much more natural than it may appear at first glance, and it is worth trying to see why.

Gödel's interpretation of implication has striking consequences. Firstly, any implication between predictions, $(\forall x, A(x)) \rightarrow (\forall u, B(u))$, where $x$ and $u$ each ranges over a special set, is to be proved by constructing a function $\bar{x}(u)$ and verifying that $\forall u, A(\bar{x}(u)) \rightarrow B(u)$. Taking the conclusion to be $0=1$, we get an interpretation of *negation*. For (not $P$) is just the assertion that $P \rightarrow (0=1)$. Thus, if $P$ has the predictive form, $\forall x, A(x)$, with $x$ ranging over a special set—e.g. the positive integers—then the Gödel interpretation of (not $P$) takes the strongly affirmative form, $\exists x_0$, not $A(x_0)$. In other words, we should expect that if we can *somehow* negate $P$ then we can actually produce a counterexample. For a general $\exists y \forall x, A(x, y)$, with $y$ and $x$ ranging over special sets, its negation here takes the form of constructing an $\bar{x}(y)$ and proving $\forall y$, not $A(\bar{x}(y), y)$. It is also instructive to compare any such incomplete statement with the Gödel interpretation of its double negation. All these interpretations fit well with the basic constructivist attitude of seeking strongly affirmative versions of negativistic concepts.

Finally, we should not forget the status of all such "guiding principles." They are definitely extramathematical observations, temporal, always subject to future modification, even rejection, whenever new developments should so require.

**Logic and meaning.** To go further, let us first go back. Constructive mathematics, beginning with Kronecker, is concerned with meaning. We may contrast this concern with the attitude of certain logicians, as expressed by the following remarks of Bertrand Russell (1901).

> Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true. . . . The proof that all pure mathematics, including

[22] This is discussed in Bishop's paper;[5] but Gödel implication is applied there without any restriction on the domains of the variables. As Bishop himself recently pointed out, this is "downright wrong," for it can produce incorrect interpretations.

> Geometry, is nothing but formal logic, is a fatal blow to the Kantian philosophy.[23]

Nevertheless, constructive mathematics starts off with the Kantian, indeed, ancient, observation that mathematics has content and meaning, independent of logical considerations. For instance, the principle of mathematical induction is correct *simply by virtue of its meaning*. For the same reason, the assertion that $0=1$ is incorrect. In fact, every constructively proved theorem (i.e. every complete theorem) is correct precisely on account of its meaning.

Moreover, as we have seen, the way constructive mathematics is directed toward predictive and descriptive assertions suggests that any incomplete statement should be interpreted as meaning, first of all, that it has a completion (gotten by actually constructing the kind of object that is asserted to exist). From this standpoint, a formally derived assertion of existence, even together with its formal proof, but without a construction, cannot even be regarded as answering "yes" to the question of whether such an object exists. This is not at all to ignore the fact that such a formal proof of existence will, nearly always, provide a verification (i.e. a completion) of some other incomplete statement which is *logically* equivalent with the original one: perhaps its double negation or, in the case of an implication, its contrapositive. But, as one can see quite vividly from simple examples, this is a different assertion, *usually with a very different meaning*. For instance, if we regard the uniform boundedness principle as asserting the existence of a certain *real number*, an upper bound for the set of norms of some family of operators on a Banach space, then its contrapositive asserts the existence of a *point in that space*, having certain properties.

Of course, classically, even if we recognize these differences in meaning, we must admit the truth of any incomplete assertion which is shown to be logically equivalent with some other complete (i.e. constructively proved) theorem. In fact, this is generally regarded as a very powerful nonconstructive way of correctly guessing the truth of existence statements, a way that enables us to concentrate on the pure form and structure of mathematics by separating out the fact of existence from the business of, somehow, actually effecting a construction.

Already Kant[24] (1781) had, in another context, called attention to

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[23] From the essay, "Mathematics and the metaphysicians," in *Mysticism and logic*, W. W. Norton and Co., New York, 1929.

[24] *Critique of pure reason*, Norman Kemp Smith's translation, Macmillan, New

the excessive "simplemindedness" of such an approach; and the discovery of the paradoxes sounded louder warnings that, even in mathematics, there might be limits to the domain of validity of Aristotelian reasoning based on excluded middle. But it remained for Brouwer[25] (1908) to show how logic and meaning were already in blatant contradiction in some of the basic assertions of classical analysis; among them, the Bolzano-Weierstrass theorem. By simply observing what these formally valid theorems said in certain specific cases, Brouwer found immediately that they could be maintained only at the price of drastically altering—and muddying—the bedrock of our mathematics, our basic concept of the positive integers. He showed that to accept such assertions entails, unavoidably, treating on a par with 1, 2, 3, $\cdots$ [26] such exotica as *the truth value* of any predictive assertion of the form, $\forall n$, $T(n)$ (where $n$ ranges over the *real* positive integers and each $T(n)$ is *really* finitely verifiable).[27]

To see how this bizarre and undesirable situation arises, consider the Bolzano-Weierstrass theorem. It says, plainly enough, that for each sequence $(r_n)$ of real numbers, with all $0 \leqq r_n \leqq 1$, there exists a strictly increasing sequence $(n_k)$ of positive integers such that, for all $k$, we have

$$\left| r_{n_{k+1}} - rn_k \right| \leqq 1/2^k.$$

If we now apply this to any specific nondecreasing sequence of 0's and 1's we find immediately that the only issue is the existence of $n_2$. For any such subsequence $(r_{n_k})$ must necessarily be constantly 0 or constantly 1 from the second term on, *depending on whether all* $r_n = 0$ *or some* $r_n = 1$. Thus the status of $r_{n_2}$ is quite different from that of $r_1, r_2, r_3, \cdots$. For, though we insist that the definition of the sequence $(r_n)$ provide a finitely performable procedure that enables us, for each specific $n$, to compute $r_n = 0$ or $r_n = 1$, the term $r_{n_2}$ given by the Bolzano-Weierstrass theorem can, as it stands, only be described as the truth value of the assertion $\forall n$, $r_n = 0$. (Note that every predictive assertion of the form $\forall n$, $T(n)$ can be translated into this

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York, 1961. On p. 262 we have "For to substitute the logical possibility of the *concept* (namely, that the concept does not contradict itself) for the transcendental possibility of *things* (namely, that an object corresponds to the concept) can deceive and leave satisfied only the simpleminded." I thank B. Mazur for showing me this reference.

25 *De onbetrouwbaarheid der logische principes.* Tijdschrift voor wijsbegeerte 2.

26 And even much larger specific integers like $10^{10^{10}}$ that are constructed recursively in terms of the general concept of integer.

27 The linguistic contortions of the parenthetical remark already indicate the way our concrete understanding of what a finite number is gets obscured.

form by taking $r_n$ to be the truth value of the finitely verifiable assertion $T(1) \wedge \cdots \wedge T(n)$.)

Thus Brouwer showed that we must choose between our *a priori* concept of the positive integers and the free use of the principle of excluded middle beyond the domain of what is known to be finitely verifiable. Brouwer opted for the former and argued that such intrinsically nonconstructive assertions as the Bolzano-Weierstrass theorem, though formally valid, should be considered to be incorrect on account of their meaning.

**The reaction to Brouwer's critique.**

> Brouwer's writings have revealed that it is illegitimate to use the principle of excluded middle in the domain of transfinite arguments. (Kolmogorov, 1925)[28]

> In any case, those logical laws that man has always used since he began to think, the very ones that Aristotle taught, do not hold. . . . logic alone does not suffice. The right to operate with the infinite can be secured only by means of the finite. (Hilbert, 1925)[29]

> Brouwer, like everyone else, required of mathematics that its theorems be (in Hilbert's terminology) "real propositions," meaningful truths. But he was the first to see exactly and in full measure how it has in fact everywhere far exceeded the limits of contentual thought. . . . In the contentual considerations that are intended to establish the consistency of formalized mathematics Hilbert fully respects these limits, and he does so as a matter of course; we are really not dealing with artificial prohibitions here by any means. (Weyl, 1927)[30]

Brouwer's observations influenced Hilbert's program of formalization in more ways than one. The original concern of Hilbert was to provide a secure framework for Cantorian mathematics, especially the theory of transfinite numbers, free from the danger of paradoxes. His scheme, derived from his earlier work on the relative consistency of geometry, was to construct, from the most primitive objects of thought, an *accurate* model of classical mathematics: one whose consistency would be determined by the most intuitive considerations, though not necessarily while standing on one leg. But Brouwer's critique, of conceded validity, posed for the formalists a new question far more serious than that of consistency. Namely, what is the point

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[28] *On the principle of excluded middle.*[1]

[29] *On the infinite.*[1]

[30] *Comments on Hilbert's second lecture on the foundations of mathematics.*[1]

of an accurate, even consistent, formalization of an incorrect theory? If we no longer discuss this point, it is not because the theory has meanwhile been corrected; but rather, as Weyl[30] explained, because Hilbert "saved" classical mathematics *"by a radical reinterpretation of its meaning* without reducing its inventory, namely, by formalizing it, thus transforming it in principle from a system of intuitive results into a game with formulas that proceeds according to fixed rules."

Weyl goes on to say that this step was necessitated by "the pressure of circumstances"; and here we arrive at the root of the matter. The circumstances were that Brouwer's critique, coming on the heels of the still unresolved crisis of Cantor's set theory, completely undermined the previously unquestioned belief that the great theories of classical mathematics—analysis, arithmetic, algebra, geometry—are a true expression of some underlying real content. The nearly universal, though, as Bishop's work shows, completely mistaken, judgment was that most of the main theorems, especially in analysis, were, after all, "merely ideal propositions" and that to accept the consequences of Brouwer's observations would mean to wreck the great theories of mathematics and permanently cripple future development. Also, the very limited reconstruction of these theories within Brouwer's own intuitionistic program[31] appeared to confirm this judgment.

> Taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists. To prohibit existence statements and the principle of excluded middle is tantamount to relinquishing the science of mathematics altogether.
> (Hilbert, 1927)[1]

> That from this point of view only a part, perhaps only a wretched part, of classical mathematics is tenable is a bitter but inevitable fact. Hilbert could not bear this mutilation.
> (Weyl, 1927)[30]

> As to the mutilation of mathematics of which you accuse me, it must be taken as an inevitable consequence of our standpoint.
> (the fictitious INT. of Heyting, 1955)[32]

> Moreover, if the intuitionistic attitude should oust the classical view it might take generations to save, and to firmly base with intuitionistic methods, those parts of mathematics which do not

[31] Plus its radical point of view and aspects of the way it was presented.

[32] *Intuitionism, an introduction*, North-Holland, Amsterdam.

become meaningless or false according to the new conceptions.
(Fraenkel & Bar-Hillel, 1958)[2]

Etc. etc. etc.

So much for "the pressure of circumstances." But in a more optimistic mood, Hilbert saw formal mathematics as a way of reaching the *real* by passing through the *ideal*. That is, one may use all the formal machinery, in particular, nonconstructive but formally valid existence statements (such as the Bolzano-Weierstrass theorem), to prove, formally, *real* propositions, i.e. predictive ones. However, not only are such considerations largely ignored or blurred nowadays, but we have already quoted, and can confirm, Bishop's observation that in practice such proofs are already constructive or can easily be made so.[33] Indeed, as Weyl[30] pointed out, the hard part is not to find constructive proofs of predictive assertions which have already been proved classically, but rather "to fill out the theorems of classical mathematics" by replacing purely formal assertions of existence by constructions.[34]

The fact is, as Bishop's book demonstrates for a sizable portion of analysis, that classical mathematics is full of significant constructive content. Furthermore, one of Bishop's main points is that, given the right general way of looking, and some groundwork, much of the job of finding it becomes routine. Of course there remain many important areas where no more than preliminary investigations have been made, e.g. the mathematics built on Hilbert's basis theorem, but there are so far no grounds for pessimism. The contrary is true—and this is certainly good news.

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The moral of this story is not the relatively boring fact that the classical system can now be "filled out," but that it is time to turn back to a systematic and realistic consideration of the meaning of

[33] There are also metamathematical results along these lines going back to Kolmogorov,[28] 1925, based on comparing a formal model of a portion of constructive mathematics with the model of classical mathematics gotten by adding to it the principle of excluded middle.

[34] This raises the question of exploring the true domain of validity of the principle of excluded middle. There is a conjecture that by means of a metaconstruction one can constructivize any classical proof of an existence statement of the form $\forall x \exists y$, $A(x, y)$ (with each $A(x, y)$ verifiable) so long as that proof can be formalized in Gödel's formal constructive system with excluded middle adjoined. This is based on empirical observation that such classical theorems tend to admit completions and on Spector's important work, "Provably recursive functionals of analysis," in *Recursive function theory*, Proc. Sympos. in Pure Math., vol. 5, Amer. Math. Soc., Providence, R. I., 1962.

our mathematics, and on this basis finally begin to realize the rich promise of a truly Kroneckerian development. By insisting[35] on admitting the principle of excluded middle, regardless of its meaning, and thereby abandoning all those natural explanations and concepts based directly on meaning, classical mathematics took a step from reality, and not into paradise.[36]

> The successful formalization of mathematics helped keep mathematics on a wrong course. The fact that space has been arithmetized loses much of its significance if space, number, and everything else are fitted into a matrix of idealism where even the positive integers have an ambiguous computational existence. . . . it took the full flowering of formalism to kill the insight into the nature of mathematics which its arithmetization could have given.[34]

Really, the only way the classical mathematician can judge for himself about the truth of what is said here is by stepping outside his system—this is not easy to do!—and then comparing what classical and constructive mathematics have to say about the phenomena and structure of the one underlying mathematics. I believe that by doing this he will discover for himself Bishop's "secret still on the point of being blabbed".[37]

GABRIEL STOLZENBERG

[35] According to Hilbert[1] "No one, though he speak with tongues of angels, will keep people from negating arbitrary assertions, forming partial judgments, or using the principle of excluded middle."

[36] Thus difficulties stemming from the fictitious existence and identity classically attributed to the elements of any set soon forced abandoning the real Cantorian set theory in favor of much less natural, and still unsatisfactory, axiomatics and formalistics. Yet, constructively, a sharpened version of the set concept is given the freest play. It is quite revealing, though beyond the scope of this exposition, to pursue further the contrasting classical and constructive standpoints on such topics as countability, uncountability, decidability, formal undecidability, consistency proofs, the role of formal systems, the structure of the line, Cantor's theory of ordinals and cardinals (constructively, the line and the plane are not equipotent—because they are not homeomorphic), the continuum hypothesis (constructively not valid), and the axiom of choice (a choice *operation* is always available constructively, though not always a choice *function*).

[37] This is the epigraph of Bishop's book, taken from Lascelles Abercrombie's *Emblems of love*, the Bodley Head, Ltd. (John Lane), London.