SELF-INTERSECTIONS IN CONTINUOUS RANDOM WALK

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An n-step random walk ($n \geqq 3$) is a sequence of n straight segments, called steps, in the plane; each step is of length 1, the first step starts at the origin and each successive step starts at the end of the previous one; every step is in random direction with uniform distribution in angle. Neglecting certain events of probability 0 we define a self-intersection as the event when for some i and j, with $1 \leqq i < j \leqq n$ and $j-i>1$, the ith and the jth step have in common exactly one point, interior to each step. Let $f(n)$ be the expected number of self-intersections; it is proved that

$$f(n) = \frac{n}{4}\sum_{p=2}^{n-1}\left(1-\frac{p}{n}\right) \Bigg[1 - \frac{4}{\pi^2}\int_0^\infty\int_0^\infty (uv)^{-1}J_0(v)\cdot[J_0^{p-1}(u-v) - J_0^{p-1}(u+v)]du\,dv$$

$$+\frac{1}{\pi^5}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^{2\pi}(uvwz)^{-1}\sum_{i=1}^{8}\epsilon_i\cos(c_i\sin\theta) \tag{1}$$

$$\cdot J_0^{p-1}((a_i^2+2\eta_i a_i b_i\cos\theta+b_i^2)^{1/2})d\theta\,du\,dv\,dw\,dz\Bigg],$$

where J_0 is the Bessel function of the first kind and zero order, and the quantities indexed by i are as given below:

(2)

i	ϵ_i	η_i	a_i	b_i	c_i
1	1	-1	$w-z$	$u-v$	$v-w$
2	1	1	$w-z$	$u-v$	$v+w$
3	-1	-1	$w-z$	$u+v$	$v+w$
4	-1	1	$w-z$	$u+v$	$v-w$
5	-1	-1	$w+z$	$u-v$	$v-w$
6	-1	1	$w+z$	$u-v$	$v+w$
7	1	-1	$w+z$	$u+v$	$v+w$
8	1	1	$w+z$	$u+v$	$v-w$.

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It is further proved that for large n we have asymptotically

$$f(n) \sim (2/\pi^2)n \log n \tag{3}$$

where $2/\pi^2$ has a simple probabilistic meaning here: it is four times the probability that a random unit step from a random point inside $x^2+y^2=4$ intersects the segment $[0, 1]$ on the x-axis.

An outline of the proof follows. Let the ith step be from (x_i, y_i) to (x_{i+1}, y_{i+1}) and let it make the angle α_i with the x-axis. The line L_i which carries it has the equation

$$(y - y_i) \cos \alpha_i - (x - x_i) \sin \alpha_i = 0$$

which we write as $L_i(x, y) = 0$. For the coordinates x_i and y_i we have

$$x_i = \sum_1^{i-1} \cos \alpha_k, \quad y_i = \sum_1^{i-1} \sin \alpha_k.$$

Since we exclude from consideration the 0-probability event that any (x_i, y_i) lies on L_j with j other than $i-1$ or i, a self-intersection occurs if and only if

$$L_i(x_j, y_j)L_i(x_{j+1}, y_{j+1}) < 0 \quad \text{and} \quad L_j(x_i, y_i)L_j(x_{i+1}, y_{i+1}) < 0.$$

Hence the total number $F = F(\alpha_1, \cdots, \alpha_n)$ of self-intersections is

$$F = \frac{1}{4} \sum_{j=3}^{n} \sum_{i=1}^{j-2} [1 - \operatorname{sgn} L_i(x_j, y_j) \operatorname{sgn} L_i(x_{j+1}, y_{j+1})] \cdot [1 - \operatorname{sgn} L_j(x_i, y_i) \operatorname{sgn} L_j(x_{i+1}, y_{i+1})]$$

where to represent the discontinuous factor sgn x we use

$$\operatorname{sgn} x = \frac{2}{\pi} \int_0^\infty u^{-1} \sin ux \, du.$$

The expected number $f(n)$ of self-intersections is

$$f(n) = (2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} F d\alpha_1 \cdots d\alpha_n,$$

the above form of F is substituted into the n-tuple integral, the order of integrations is changed, and the angle integrations are carried out, leading after some lengthy transformations to (1). For the proof of (3) we use the standard asymptotics of the Bessel functions, representing $J_0^p(x)$ as $\exp(-px^2/4)$ times a power series.

It is expected that the detailed proofs, the numerical work, and further related material will appear elsewhere.

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