grams and algebraic theory is completely lacking. But the algebra *is* elegant and polished, and one hopes that one day Eilenberg and Elgot will give us the book promised in their preface.

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Piecewise Linear Topology, by J. F. P. Hudson, Benjamin, New York, 1969.

This volume consists of lecture notes from a course given by Professor Hudson at the University of Chicago in 1966–1967. His intent is "to develop PL theory from basic principles . . .". There are not many methods available in piecewise linear topology which are "elementary" in the sense that they do not use bundle theories; and as applied to manifolds these methods seem by now to have been pushed to their limits. Hudson has not proved the strongest theorems possible, but he has demonstrated thoroughly these elementary methods and their use. His treatment draws heavily on Zeeman's

I.H.E.S. Seminar notes "Combinatorial Topology" of 1963. He has restricted attention mostly to manifolds, but has added the many details of proof that Zeeman slid over and also many more results, some of which have not appeared in print before. Thus "Piecewise Linear Topology" is likely to be a standard text for a long time.

In Chapter I a "Euclidean polyhedron" is defined as the union of finitely many convex linear cells in some Euclidean space; and then (following Zeeman) a "piecewise linear structure" on a topological space X is defined by means of local "coordinate maps" from Euclidean polyhedra to X. It is proved that Euclidean polyhedra and piecewise linear maps between them can be triangulated by finite simplicial complexes and simplicial maps. A "piecewise linear manifold" has a mixed definition: it is a Euclidean polyhedron in which every point has a coordinate neighbourhood which is a piecewise linear ball. One of Hudson's improvements on Zeeman's treatment is to introduce early the cell complex "dual" (in the sense of Poincaré) to a simplicial triangulation of a manifold. It is now easy to prove that the boundary of a manifold has a collar neighbourhood in the manifold. Then follows M. Cohen's simple proof that the complement of an n-ball in an n-sphere is an n-ball. This in turn greatly simplifies Zeeman's proof of the uniqueness theorem for regular neighbourhoods.

In Chapter II Hudson defines "collapsing" and proves that if X is a subpolyhedron of a manifold M, then a second-derived neighbourhood of X in M collapses to X. Following Zeeman, a "regular neighbourhood" of X in M is a closed neighbourhood of X in M which is a manifold and which collapses to X. (I prefer Cohen's equivalent definition: a "regular neighbourhood" of X in M is anything which can be described as a second-derived neighbourhood of X in M; for this, and not Hudson's definition, tells one how to define regular neighbourhoods in the general context of piecewise linear topology, when M is not a manifold.) Then comes the main theorem: any regular neighbourhoods N_1 and N_2 of X in M are piecewise linearly homeomorphic keeping X fixed. An important improvement on Zeeman's version is: if N_1 and N_2 meet the boundary of M "regularly," then this homeomorphism can be realized by an isotopy of M keeping X fixed.

In Chapter III a "PL space" is defined to be a Hausdorff topological space with a piecewise linear structure, and the results of the previous chapters are briefly extended to PL spaces. It is proved that PL spaces can be thought of as locally finite, but possibly infinite, simplicial complexes. Hudson has omitted Zeeman's more general definition of a "polyspace," which envisaged the study of piecewise linear function spaces, since they have never been applied to the study of manifolds (to my knowledge).

In differential topology general position and transversality theorems are proved using differentiable function spaces. In piecewise linear topology we have had to be content with a weaker procedure: given a simplicial triangulation of Euclidean space one shifts the vertices slightly one at a time; this can be done so that the induced shift of Euclidean space puts given subspaces into "general position." In a general piecewise linear manifold one takes Euclidean coordinate patches and performs the previous construction in each one. In Chapter IV Hudson illustrates methods of this nature. The basic program is: first to approximate a continuous map $f:P\rightarrow Q$ by a piecewise linear map f_1 homotopic to f. Then if Q is a manifold one can approximate f_1 by a "nondegenerate" map f_2 via a piecewise linear homotopy ("nondegenerate" means: $f_2^{-1}(q)$ is a finite set for all q in Q). Finally one can approximate f_2 by f_3 such that the r-fold singular set of f_3 (that is: $\{p$ in P such that $f_3^{-1}(f_3(p))$ has at least r points$\}$) is of dimension as small as is generically possible, namely $r\cdot$(dimension of P) $-(r-1)\cdot$(dimension of Q) (N.B. there is a mistaken sign in the definition on page 90). Such an f_3 is said to be in "general position." Hudson proves more complicated results, but Armstrong and Zeeman's more delicate notion of "transimpliciality" (which yields a form of transversality) is not mentioned.

Chapter V proves the theorem that distinguishes piecewise linear topology from differential topology: that sphere pairs and proper ball pairs of codimension at least three are unknotted; that is, piecewise linearly homeomorphic to the standard pairs. Hudson follows Zeeman's method of "sunny collapsing," for which a simple general position argument is required. Lickorish's generalization is stated without proof: that any proper subcone of a ball unknots in codimension at least three.

In Chapter VI Hudson defines when two embeddings of a manifold M in another manifold Q are "concordant," "isotopic," "ambient isotopic" and "isotopic by moves." This chapter proves the last three equivalent; the remaining equivalence is proved in Chapter IX. A special case in which concordance implies isotopy is, however, needed: the uniqueness of compatible collar neighbourhoods of (boundary Q, boundary M) in (Q, M). (The existence of such collars is easily proved using dual cells, as in Chapter I.) The main theorem is that if dim $Q\geqq$dim $M+3$, then any proper embedding $f:M\times I\rightarrow Q\times I$ which commutes with the projection on I can be extended to a piecewise linear homeomorphism of $Q\times I$ with the same property. Hudson also proves the n-isotopy theorem, in which the interval I of the main theorem is replaced by I^n, the n-cube. This is an important result whose proof has not appeared before.

Chapter VII gives some engulfing theorems; one needs general position but not the results of Chapters V and VI. There are two types of engulfing theorem: one for engulfing a subpolyhedron X of a manifold Q by an open subset; and the other for engulfing X from a subpolyhedron Y of Q by finding a $Y'\subseteq Q$ which collapses to Y, which contains X and which is of dimension at most dim $X+1$. Hudson simplifies Zeeman's treatment by assuming (roughly) that Q is without boundary. The first type of engulfing is used to prove Stallings' theorem: for $n\geqq 5$, Euclidean n-space is piecewise linearly characterized as an $(n-3)$-connected manifold which is 1-connected at infinity. Also a weak form of the generalized Poincaré conjecture: that for $n\geqq 5$, a piecewise linear n-manifold homotopy equivalent to the n-sphere is *topologically* homeomorphic to the n-sphere.

Chapter VIII uses engulfing to prove embedding theorems. For example: assume given manifolds M^m and Q^q, $q\geqq m+3$, and $f: M\rightarrow Q$ a proper continuous map whose restriction f: boundary $M\rightarrow$boundary Q is a piecewise linear embedding. Assume M and Q are "double-point" connected (that is, roughly $(2m-q)$-connected); then f is homotopic to a piecewise linear embedding, keeping f fixed on boundary M. Irwin's theorem is also proved. Hudson's own generalization is not mentioned, however: one need only require the map f to be double-point connected, and M and Q to be triple-point connected. It is true that a stronger theorem (in which no triple-point assumptions need be made) has been proved by Browder, Casson and Sullivan using bundle-theoretic methods; a version of this theorem is stated at the beginning of Chapter XII without even the comment that no proof follows. But I think that a simple version of Hudson's theorem is not beyond the scope of this book and would introduce the method of "surgery" on the singular set—a sort of piecewise linear version of Haefliger's embedding technique in the differentiable category. A preliminary version of this method is used by Zeeman in his chapter on engulfing.

Chapter IX gives the first published proof that concordance implies isotopy: if two embeddings f_0, f_1 of an m-manifold M in a q-manifold Q are "concordant"—that is, there is an embedding $F: M\times I\rightarrow Q\times I$ which restricts to $f_0\times 0$ on $M\times 0$ and to $f_1\times 1$ on $M\times 1$—and if $q\geqq m+3$, then f_0 and f_1 are isotopic—that is, one can choose F to be level-preserving.

In Chapter X the theorem that concordance implies isotopy is used in combination with embedding theorems of Chapter VIII to give quick proofs of analogous unknotting theorems. If f, $g: M \to Q$ are embeddings of one manifold in another which are homotopic, then under conditions slightly stronger than those of Chapter VIII f and g are isotopic.

Chapter XI describes the first Shapiro-Wu obstruction to homotoping a map $f: M^m \to Q^q$ to an embedding. This obstruction lies in $H_{2m-q}(M; Z_2)$, where $2m-q$ is the double-point dimension. There is a similar obstruction in $H_{2m-q+1}(M; Z_2)$ to turning a homotopy between two embeddings into an isotopy. In case Q is Euclidean space, M is $(2m-q)$-connected, and one embedding $f_0: M \to Q$ is fixed, then *every* element of $H_{2m-q+1}(M; Z_2)$ occurs as the obstruction to isotoping some embedding of M in Q to f_0. Stronger results are stated in special cases, such as when this function from isotopy classes of embeddings to the homology group is one-to-one, and when one can use $H_{2m-q+1}(M; Z)$ instead.

Chapter XII states the Browder, Casson and Sullivan theorem: if $f: M^m \to Q^q$ is a homotopy equivalence of manifolds, $q \geqq m+3$, M is without boundary, and $\pi_1(\text{boundary } Q) \approx \pi_1(Q)$, then f is homotopic to an embedding. A related theorem of Stallings is proved: if $f: K^m \to Q^q$ is a homotopy equivalence, K a finite complex, Q a manifold, and if f is $(2m-q+1)$-connected, then there is a subcomplex $K' \subseteq Q$ such that dim $K' \leqq m$, $f(K) \subseteq K'$ and $f: K \to K'$ is a simple homotopy equivalence. (These results together imply the embedding theorem similar to those of Chapter VIII that I mentioned.)

The final section of the book uses some of everything to prove the s-cobordism theorem. The treatment is standard, but apart from Milnor's "Lectures on the h-cobordism theorem" is the only version that I know of to be published in book form. In piecewise linear topology one cannot use vector fields, as Milnor does; but one has a handle decomposition of a piecewise linear manifold readily available: it is provided by the dual cell complex to a simplicial triangulation of the manifold. Perhaps in order to increase the difference from Milnor's treatment, Hudson includes a discussion of Whitehead torsion (condensed from Milnor's article in this *Bulletin*, 1966) sufficient to prove the s-cobordism theorem and to show that h-cobordisms with a given manifold M of dimension at least five are classified precisely by the Whitehead group $\mathrm{Wh}(\pi_1(M))$.

As mentioned, "Piecewise linear topology" is the result of a year-long graduate course. It can be studied after a one-semester course in algebraic topology which covers homology theory and homotopy theory up to, say, Whitehead's theorem. At the time of writing, Hudson's is the *only* book on piecewise linear topology that is readily available (compared to Zeeman's notes, Stallings' notes or Glaser's part-published book); and it is not expensive. The proofs are careful and complete—I have found very few misprints and mistakes. Thus this book can be used by a student to teach himself the subject, as Zeeman's notes have been used. However, its consequent length and density make it likely that only the most devoted and determined graduate student will do so. Moreover there are not enough diagrams, except in the section on the s-cobordism theorem, and virtually no exercises or examples. In particular the truth of the generalized Poincaré conjecture is not mentioned as a corollary of the s-cobordism theorem. The sections of Zeeman's notes that I have found most interesting are his "digressions," on the dunce-hat and on linked spheres in Euclidean space, for example. They are lacking in Hudson. But they can be found from his very good bibliography, which is not confined to piecewise linear topology, but lists papers in "topological" topology, differential topology and bundle-theories. It is unfortunate that this bibliography is not referred to in the text, so that the self-teaching student may have difficulty choosing outside reading. Thus Hudson's book is likely to be most used as a text for graduate courses. In this role I think it will be extremely useful, and I expect that for years to come instructors and students alike will say: "For details of proof—see Hudson."

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