BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 4, July 1971

SUBSTITUTION MINIMAL FLOWS[1]

BY JOHN C. MARTIN

Communicated by W. Gottschalk, February 25, 1971

We investigate the structure of a certain class of minimal symbolic flows (substitution minimal flows) which are natural generalizations of the widely studied Morse minimal set (see, for example, [**3**], [**5**]). We present here a brief description of the major results; detailed proofs will appear elsewhere. The author wishes to thank William Veech for his help in the preparation of this paper.

Let $S = \{0, 1, \cdots, b-1\}$, and for $n \geqq 1$, let $S^n = \{f: \{0, 1, \cdots, n-1\} \to S\}$. If $A \in S^n$, we represent A as $a_0 \cdots a_{n-1}$, where $a_i = A(i)$; we refer to A as an n-block (over S). For $A \in S^n$, $B \in S^m$, we let $AB = a_0 a_1 \cdots a_{n-1} b_1 b_2 \cdots b_{m-1}$, so that $AB \in S^{n+m}$. A *substitution* θ $(=\theta^1)$ of length r over S is a map $\theta: S \to S^r$ with $\theta(0)(0) = 0$. For $k \geqq 2$, if $\theta(j) = a_0 a_1 \cdots a_{r-1}$, we define $\theta^k(j) = \theta^{k-1}(a_0) \cdots \theta^{k-1}(a_{r-1})$. We define a sequence x'_θ over S by letting the r^k-block $x'_\theta(0)x'_\theta(1) \cdots x'_\theta(r^k - 1)$ be $\theta^k(0)$, for each $k \geqq 1$. θ is an *admissible* substitution if θ is one-to-one, range $x'_\theta = S$, and x'_θ is a recurrent, nonperiodic sequence. (It is not difficult to prescribe simple conditions which ensure that θ is admissible.) θ is *simple*, if for $i, j \in S$ $(i \neq j)$, $\theta(i)(n) \neq \theta(j)(n)$ $(0 \leqq n \leqq r-1)$. If θ is an admissible substitution, we choose any recurrent extension x_θ of x'_θ to the integers, and we define $\mathfrak{X}_\theta = (X_\theta, T)$ to be the flow whose phase space X_θ is the orbit-closure of x_θ under the left shift T, in the space of all doubly infinite sequences over S (with the product topology). X_θ is an infinite, compact metric space, and $\mathfrak{X}_\theta$ is a minimal flow. Finally, we obtain a positive integer $m(\theta)$ with $\gcd(m(\theta), r) = 1$ so that S is partitioned into nonempty sets $S_0, S_1, \cdots, S_{m(\theta)-1}$, and if $i \in S_{n(i)}$ $(i \in S)$, the sequence of integers $n(x_\theta(j))$ $(j = 0, 1, \cdots)$ is periodic of period $m(\theta)$.

If θ is a fixed admissible substitution of length r over S, our principal results may be stated as follows. Some of our results generalize certain results in [**1**] and [**4**]. (All definitions are as in [**10**].)

AMS 1969 *subject classifications.* Primary 5420.

Key words and phrases. Symbolic flow, Morse minimal set, substitution, equicontinuous structure relation, almost automorphic flow, almost automorphic extension, isometric extension, AI extension, AI flow, point-distal flow.

[1] Research partially supported by NSF Grant No. 8997.

THEOREM 1. *$\mathfrak{X}_\theta$ is a point-distal flow with a residual set of distal points.*

THEOREM 2. *Let Σ be the equicontinuous structure relation on X_θ. Then $\mathfrak{X}_\theta/\Sigma$ is isomorphic to the equicontinuous flow $(Z_{m(\theta)}\times Z^r, T)$, where $Z_{m(\theta)}$ is the cyclic group of order $m(\theta)$, Z^r is the r-adic completion of the integers, and T is the homeomorphism determined by addition of the group element* (1, 1).

COROLLARY. *If θ is a binary substitution, $\mathfrak{X}_\theta/\Sigma = (Z^r, T)$.*

THEOREM 3. *$\mathfrak{X}_\theta$ is an almost automorphic flow if and only if there exist integers i, j, k $(0 \leqq i \leqq m(\theta)-1, j \geqq 1, 0 \leqq k \leqq r^j-1)$ with $\theta^j(p)(k) = \theta^j(q)(k)$ for $p, q \in S_i$.*

In [8], Veech represents the Morse flow (the substitution flow generated by the binary substitution $\theta(0)=01$, $\theta(1)=10$) as an isometric extension of an almost automorphic extension of (Z^2, T). This may be generalized in the following manner. We define $P_\theta = \{x_\theta(j)x_\theta(j+1): j=0, 1, \cdots\} \subset S^2$; $A_{ijk} = \{\theta^j(p)(k)\theta^j(p)(k+1): p \in S_i\} \subset P_\theta$ $(0 \leqq i \leqq m(\theta)-1, j \geqq 1, 0 \leqq k \leqq r^j-2)$.

THEOREM 4. *If θ is simple, $\mathfrak{X}_\theta$ is an* AI *extension (i.e., an isometric extension of an almost automorphic extension) of an equicontinuous flow if and only if the collection $\{A_{ijk}\}$ is a partition of P_θ.*

It can easily be seen that this condition holds automatically for every simple binary substitution. We obtain

THEOREM 5. *If θ is a binary substitution of length r, $\mathfrak{X}_\theta$ is either an almost automorphic flow or an* AI *extension of the equicontinuous flow (Z^r, T).*

THEOREM 6. *If θ is simple, and r and b are both prime, $\mathfrak{X}_\theta$ is an* AI *flow if and only if the collection $\{A_{ijk}\}$ is a partition of P_θ.*

By Theorem 6, we obtain a class of point-distal flows with a residual set of distal points which are not AI flows. This is significant in the light of Veech's structure theorem for point-distal flows [10], according to which every point-distal flow with a residual set of distal points has an almost automorphic extension which *is* an AI flow. (Leonard Shapiro, in [6], has constructed examples, of a different sort, of point-distal, non-AI flows.)

EXAMPLE. Let $b=r=3$; $\theta(0)=011$, $\theta(1)=202$, $\theta(2)=120$. It can be easily verified that θ is admissible and simple and that $m(\theta)=1$. We have $A_{010} \cap A_{011} = \{20\}$, and thus, by Theorem 6, $\mathfrak{X}_\theta$ is not an AI flow.

We remark that for substitutions of nonconstant length (i.e., if the blocks $\theta(0), \theta(1), \cdots, \theta(b-1)$ are not of the same length), the situation is substantially different. $\mathfrak{X}_\theta$ is no longer point-distal in general, and for certain θ, $\mathfrak{X}_\theta$ can be shown to be weakly mixing. We hope to discuss this at greater length in a later paper.

REFERENCES

1. W. H. Gottschalk, *Substitution minimal sets*, Trans. Amer. Math. Soc. **109** (1963), 467–491. MR **32** #8325.

2. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloq. Publ., vol. 36, Amer. Math. Soc., Providence, R. I., 1955. MR **17**, 650.

3. M. Keane, *Generalized Morse sequences*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **10** (1968), 335–353. MR **39** #406.

4. Harvey B. Keynes, *The proximal relation in a class of substitution minimal sets*, Math. Systems Theory **1** (1967), 165–174. MR **35** #997.

5. M. Morse, *Recurrent geodesics on a surface of negative curvature*, Trans. Amer. Math. Soc. **22** (1921), 84–100.

6. Leonard Shapiro, *Distal and proximal extensions of minimal flows* (preprint).

7. W. A. Veech, *Almost automorphic functions on groups*, Amer. J. Math. **87** (1965), 719–751. MR **32** #4469.

8. ———, *Strict ergodicity in zero dimensional dynamical systems and the Kronecker-Weyl theorem* mod 2, Trans. Amer. Math. Soc. **140** (1969), 1–33. MR **39** #1410.

9. ———, *Minimal transformation groups with distal points*, Bull. Amer. Math. Soc. **75** (1969), 481–486.

10. ———, *Point-distal flows*, Amer. J. Math. **92** (1970), 205–242.

RICE UNIVERSITY, HOUSTON, TEXAS 77001