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INTEGRATION OF COMPLEX VECTOR FIELDS[1,2]

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1. **The problem.** First we describe the local problem. Let $L_1, \ldots, L_m$ be vector fields defined in a neighborhood U of the origin in $\boldsymbol{R}^n$ by the expressions:

$$L_k = \sum_{j=1}^{n} a_k^j \frac{\partial}{\partial x_j}, \qquad k = 1, \ldots, m, \tag{1.1}$$

where the a_k^j are infinitely differentiable complex-valued functions on U. The local problem is to "solve" the equations

$$L_k u = f_k, \qquad k = 1, \ldots, m. \tag{1.2}$$

That is, given functions $f_1, \ldots, f_m$ we wish to find conditions for the existence of a function u satisfying (1.2); further we wish to describe the set of functions satisfying (1.2) and also their dependence on the f_k, especially with respect to regularity properties. First consider the homogeneous case when $f_k = 0$, i.e.,

$$L_k u = 0, \qquad k = 1, \ldots, m. \tag{1.3}$$

Any function u satisfying all the above equations must also satisfy the equations

$$[L_k, L_h]u = L_k L_h u - L_h L_k u = 0. \tag{1.4}$$

Thus it is reasonable to assume that the space spanned by the vector fields $L_1, \ldots, L_m$ is closed under the bracket operation.

Condition A. This condition is satisfied if

$$[L_k, L_h] = \sum_j a_{kh}^j L_j \tag{1.5}$$

where the $a_{kh}^j \in C^\infty(U)$.

From (1.2) we obtain

$$[L_k, L_h]u = L_k f_h - L_h f_k$$

and hence (1.5) yields

$$L_k f_h - L_h f_k = \sum_j a_{kh}^j f_j, \qquad 1 \leqq k < h \leqq m. \tag{1.6}$$

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Thus we have established the following lemma.

1.7 LEMMA. *If* $L_1, \ldots, L_m$ *satisfy condition* A *and if there exists a function* u *satisfying* (1.2) *then* $f_1, \ldots, f_m$ *satisfy the system* (1.6).

Another obvious necessary condition is given in the following.

1.8 LEMMA. *If* (1.2) *is satisfied in* U *and if* $P \in U$ *and*

$$\sum_k c^k a_k^j(P) = 0, \quad \text{for } j = 1, \ldots, n,$$

with $c^k \in \boldsymbol{C}$ *then it follows that*

$$\sum c^k f_k(P) = 0. \tag{1.9}$$

We will assume the following condition which implies (1.9).

Condition B. This condition is satisfied if $L_1, \ldots, L_m$ are linearly independent at each point, i.e. if the matrix a_k^j is of rank m at each point.

As will be seen later condition B is extremely restrictive but as yet it has not been possible to weaken it.

The global problem may be formulated as follows. Let $\boldsymbol{C}T(U)$ denote the complexified tangent bundle over U and let $\boldsymbol{L}$ be a complex subbundle of $\boldsymbol{C}T(U)$ with fiber of complex dimension m. Condition A can now be replaced by requiring that if $L, L' \in \Gamma(\boldsymbol{L}, V)$ (i.e. sections of $\boldsymbol{L}$ over $V \subset U$) then

$$[L, L'] \in \Gamma(\boldsymbol{L}, V). \tag{1.10}$$

We denote by $\boldsymbol{L}^*$ the dual bundle of $\boldsymbol{L}$ and for every open $V \subset U$ we define the map

$$\mathscr{D} : C^\infty(V) \to \Gamma(\boldsymbol{L}^*, V)$$

if $u \in C^\infty(V)$, $\mathscr{D}u$ is determined by setting

$$\langle \mathscr{D}u, L \rangle = L(u) \tag{1.11}$$

for all $L \in \Gamma(\boldsymbol{L}, V)$ where $\langle\ ,\ \rangle$ denotes the pairings between the fibers of $\boldsymbol{L}^*$ and $\boldsymbol{L}$. Equation (1.2) can now be written as follows:

$$\mathscr{D}u = f \tag{1.12}$$

where $u \in C^\infty(U)$ and $f \in \Gamma(\boldsymbol{L}^*, U)$. Choosing $L_1, \ldots, L_m$ to be a local basis for $\Gamma(\boldsymbol{L}, U)$ and $\omega^1, \ldots, \omega^m$ to be the dual bases of $\Gamma(\boldsymbol{L}^*, U)$, it is clear that (1.2) expresses (1.12) in terms of this local basis. The advantage of the above formulation is that it makes sense whenever U is a C^∞ manifold. In order to reformulate Lemma 1.7 in those terms, let $\boldsymbol{L}^* \wedge \boldsymbol{L}^*$ denote the skew symmetric product of $\boldsymbol{L}^*$ with itself. We then have the natural isomorphism $\boldsymbol{L}^* \wedge \boldsymbol{L}^* \approx (\boldsymbol{L} \wedge \boldsymbol{L})^*$. Define the map $\mathscr{D}: \Gamma(\boldsymbol{L}^*, V) \to \Gamma(\boldsymbol{L}^* \wedge \boldsymbol{L}^*, V)$ by setting

$$\langle \mathscr{D}f, L \wedge L' \rangle = L(\langle L', f \rangle) - L'(\langle L, f \rangle) - \langle [L, L'], f \rangle, \tag{1.13}$$

where $f \in \Gamma(\boldsymbol{L}^*, V)$ and $L, L' \in \Gamma(\boldsymbol{L}, V)$; the symbol $\langle\, , \rangle$ denotes the pairing between the fibers of $\boldsymbol{L} \wedge \boldsymbol{L}$ and $\boldsymbol{L}^* \wedge \boldsymbol{L}^*$. Thus we obtain the complex

$$C_0^\infty(V) \xrightarrow{\mathscr{D}} \Gamma(\boldsymbol{L}^*, V) \xrightarrow{\mathscr{D}} \Gamma(\boldsymbol{L}^* \wedge \boldsymbol{L}^*, V); \tag{1.14}$$

it is easily checked that

$$\mathscr{D}^2 = 0. \tag{1.15}$$

Lemma 1.7 can now be reformulated as follows. A necessary condition for the existence of u satisfying (1.12) is

$$\mathscr{D}f = 0. \tag{1.16}$$

2. **Examples.** The following are examples of manifolds M on which we distinguish a subbundle $\boldsymbol{L}$ of the complexified tangent bundle $\boldsymbol{CT}$, which satisfies conditions A and B and we discuss the corresponding local and global problems of solving (1.12).

(I) M is any differentiable manifold and $\boldsymbol{L} = \boldsymbol{CT}$. In this case $\mathscr{D}$ is the exterior derivative (usually denoted by d). The solution of the local problem is classical, usually called the Poincare lemma, and the global problem can be analyzed by means of elliptic boundary-value problems (see [**2**] and [**3**]).

(II) M is a complex manifold and let $\boldsymbol{L}$ be the bundle of tangent vectors of type (0, 1) (i.e. locally a section of $\boldsymbol{L}$ is of the form $\sum a^j \, \partial/\partial \bar{z}_j$ where $z_1, \dots, z_n$ are holomorphic local coordinates). In this example the operator $\mathscr{D}$ is usually denoted by $\bar{\partial}$ and the local solution is given by the so-called Dolbeault lemma. The global problem for compact manifolds can easily be reduced to determined elliptic systems; this is the so-called Hodge theory (see for example [**12**]). On a manifold with a smooth boundary the global problem is in general not solvable in the sense that the range of $\bar{\partial}$ is not closed in any of the topologies of interest; however, if certain conditions are imposed on the boundary the problem is solvable (see [**13**] and [**7**]). Some important questions about the dependence of the solution on f in Hölder norms still remain unanswered although great progress in this area has been made recently (see [**11**]).

(III) M is a differentiable manifold and $\boldsymbol{L}$ has the property that $\bar{\boldsymbol{L}} = \boldsymbol{L}$ (i.e. locally each $\bar{L}_k = \sum \bar{a}_k^j \, \partial/\partial x^j$ is a linear combination of $L_1, \dots, L_n$). Then locally by the classical Frobenius theorem we can find local coordinates $x_1, \dots, x_n$ such that a local basis for the sections of $\boldsymbol{L}$ over a coordinate neighborhood is given by $\partial/\partial x_1, \ldots, \partial/\partial x_m$ and so the local problem can be settled by an easy adaptation of the solution of example (I). However the global problem is not stable and will not be considered here (see [**28**]).

(IV) Let M_1 and M_2 be manifolds with $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ the corresponding subbundles of $\boldsymbol{C}T(M_1)$ and $\boldsymbol{C}T(M_2)$ satisfying conditions A and B. Let $M = M_1 \times M_2$ and denote by $\pi_i : M \to M_i$, $i = 1, 2$, the projection maps; these induce maps $\pi_i^* : \boldsymbol{C}T(M) \to \boldsymbol{C}T(M_i)$. Now we define $\boldsymbol{L}$ by

$$\boldsymbol{L} = \{L \in \boldsymbol{C}T(M) | \pi_i^* L \in \boldsymbol{L}_i, i = 1, 2\}. \tag{2.1}$$

This $\boldsymbol{L}$ then satisfies conditions A and B.

(V) Let M be a closed differentiable submanifold of a manifold M_1. Suppose that $\boldsymbol{L}_1 \subset \boldsymbol{C}T(M_1)$ satisfies A and B. The identity map $I : M \to M_1$ induces the map $I^* : \boldsymbol{C}T(M) \to \boldsymbol{C}T(M_1)$ and we define

$$\boldsymbol{L} = \{L \in \boldsymbol{C}T(M) | I^* L \in \boldsymbol{C}T(M_1)\}. \tag{2.2}$$

Now if we assume that $\boldsymbol{L}$ is a bundle (i.e. that the dimension of the fiber is constant) then we obtain another example. We say that $\boldsymbol{L}$ is the *restriction* of $\boldsymbol{L}_1$ to M.

The case, where M is a closed submanifold of a complex manifold M_1 and $\boldsymbol{L}_1$ is as in (II), has been studied extensively (see [**20**], [**18**], [**29**] and [**24**]). In fact if we take $M_1 = \boldsymbol{C}^2$ and

$$M = \{(z_1, z_2) \in \boldsymbol{C} | \operatorname{Re} z_2 = |z_1|^2\},$$

setting $t = \operatorname{Im} z_2$ and $z = z_1$ we find that z, t are local coordinates in the real three-dimensional manifold M. In terms of these coordinates the equation (1.12) becomes

$$\frac{\partial u}{\partial \bar{z}} + iz \frac{\partial u}{\partial t} = f. \tag{2.3}$$

This is the famous equation of H. Lewy which in general has no solution (see [**21**]). The discovery of this equation has led to several important developments in partial differential equations (see [**9**] and [**26**]). The case where $M = \{z \in \boldsymbol{C}^n | \, |z| = 1\}$ has been studied in great detail by G. B. Folland (see [**4**]).

Condition C. We say that the subbundle $\boldsymbol{L} \subset \boldsymbol{C}T$ satisfies this condition if $\boldsymbol{L} \wedge \bar{\boldsymbol{L}}$ is also a subbundle, i.e. if the fibers are of constant dimension.

It is now natural to ask whether any $\boldsymbol{L} \subset \boldsymbol{C}T$ satisfying conditions A, B and C can be realized locally performing the constructions in IV and V on the examples II and III. In case $\boldsymbol{L}$ is one dimensional this reduces to a problem of H. Lewy; namely, given a single nonvanishing complex vector field L do there exist nontrivial solutions of $Lu = 0$. In case $\boldsymbol{L}$ is real analytic (i.e. there exist coordinates such that the real and imaginary parts of the a_k^j in (1.1) are analytic) the conjecture is answered affirmatively by the

Cauchy-Kowalevsky method (see [**1**] where it is also shown that if coordinates $x_1, \ldots, x_n$ are chosen so that $\partial/\partial x_1, \ldots, \partial/\partial x_p$ span the local sections of $\boldsymbol{L} \cap \bar{\boldsymbol{L}}$ then it suffices to assume analyticity in the variables $x_{p+1}, \ldots, x_n$). Another example in which the above conjecture is established is the case when the sections of $\boldsymbol{L} + \bar{\boldsymbol{L}}$ are closed under the Lie bracket, the result is known as the Nirenberg-Frobenius theorem (see [**23**], [**25**] and [**8**]).

3. **The Laplacian and the energy form.** We will assume that $\boldsymbol{L} \subset \boldsymbol{C}T(M)$ satisfies conditions A, B and C. Denote by n the dimension of M, m the fiber dimension of $\boldsymbol{L}$ and p the fiber dimension of $\boldsymbol{L} \cap \bar{\boldsymbol{L}}$. Let $\boldsymbol{L}_0$ be a subbundle of $\boldsymbol{L}$ such that

$$\boldsymbol{L} = \boldsymbol{L}_0 \oplus \boldsymbol{L} \cap \bar{\boldsymbol{L}} \tag{3.1}$$

and let $\boldsymbol{N}$ be a subbundle of $\boldsymbol{C}T$ such that

$$\boldsymbol{C}T = \boldsymbol{N} \oplus (\boldsymbol{L} + \bar{\boldsymbol{L}}). \tag{3.2}$$

Then we have

$$\boldsymbol{C}T = \boldsymbol{N} \oplus \boldsymbol{L}_0 \oplus \bar{\boldsymbol{L}}_0 \oplus \boldsymbol{L} \cap \boldsymbol{L}. \tag{3.3}$$

Choose hermitian metrics on $\boldsymbol{N}, \boldsymbol{L}_0$ and $\boldsymbol{L} \cap \bar{\boldsymbol{L}}$; we define a hermitian metric on $\bar{\boldsymbol{L}}_0$ by setting

$$\langle u, v \rangle = \overline{\langle \bar{u}, \bar{v} \rangle} \tag{3.4}$$

where u, v are tangent vectors (at the same point) lying in $\bar{\boldsymbol{L}}_0$; thus $\bar{u}, \bar{v} \in \boldsymbol{L}_0$ and so the right-hand side has meaning. Now we define a hermitian metric on $\boldsymbol{C}T$ by making the direct sum in (3.3) orthogonal—this induces a hermitian metric on $\boldsymbol{C}T^*$, on $\boldsymbol{L}^*$ and on $\boldsymbol{L}^* \wedge \boldsymbol{L}^*$. Assume that M has a volume element dV. We define the L_2-inner product on the sections of these bundles by setting

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle \, dV. \tag{3.5}$$

Thus we have the adjoint of (1.14) (with $V = M$) which is the differential complex

$$\Gamma(\boldsymbol{L}^* \wedge \boldsymbol{L}^*, M) \xrightarrow{\mathscr{D}^*} \Gamma(\boldsymbol{L}^*, M) \xrightarrow{\mathscr{D}^*} C^\infty(M) \tag{3.6}$$

where the $\mathscr{D}^*$ are the formal adjoints of the $\mathscr{D}$ determined by requiring that

$$(\mathscr{D}^* \varphi, \psi) = (\varphi, \mathscr{D}\psi) \tag{3.7}$$

where $\varphi \in \Gamma(\boldsymbol{L}^* \wedge \boldsymbol{L}^*, M)$ and $\psi \in \Gamma(\boldsymbol{L}^*, M)$ and ψ has compact support. The map $\mathscr{D}^*: \Gamma(\boldsymbol{L}^*, M) \to C^\infty(M)$ is defined similarly and thus we have

$$(\mathscr{D}^*)^2 = 0. \tag{3.8}$$

We now define the map $\mathscr{L}:\Gamma(\boldsymbol{L}^*, M) \to \Gamma(\boldsymbol{L}^*, M)$ by setting

$$\mathscr{L}\varphi = \mathscr{D}\mathscr{D}^*\varphi + \mathscr{D}^*\mathscr{D}\varphi. \tag{3.9}$$

$\mathscr{L}$ is called the *Laplacian* of the complex (1.14), in the case of example (I) this is the usual Laplace-Beltrami operator on 1-forms.

Suppose that M is compact. Now observe that given $f \in \Gamma(\boldsymbol{L}^*, M)$ if we wish to solve the equation (1.12), then f must satisfy not only (1.16) but also, f has to be orthogonal to $\mathscr{N}(\mathscr{D}^*)$, the null space of $\mathscr{D}^*$. Since if $\psi \in \mathscr{N}(\mathscr{D}^*)$ we have

$$(f, \psi) = (\mathscr{D}u, \psi) = (u, \mathscr{D}^*\psi) = 0. \tag{3.10}$$

We denote by $\mathscr{H}$ the null space of $\mathscr{L}$. It then follows that

$$\mathscr{H} = \{\varphi \in \Gamma(\boldsymbol{L}^*, M) | \mathscr{D}\varphi = 0 \text{ and } \mathscr{D}^*\varphi = 0\} \tag{3.11}$$

since

$$(\mathscr{L}\varphi, \varphi) = \|\mathscr{D}\varphi\|^2 + \|\mathscr{D}^*\varphi\|^2, \tag{3.12}$$

and we obtain:

3.13 Lemma. *Given $f \in \Gamma(M, \boldsymbol{L}^*)$ if there exists $u \in C^\infty(M)$ satisfying* (1.12) *then f satisfies* (1.16) *and is orthogonal to $\mathscr{H}$. Furthermore the conditions* (1.16) *and orthogonality to $\mathscr{H}$ are equivalent to* (1.16) *and orthogonality to $\mathscr{N}(\mathscr{D}^*)$.*

Consider the equation

$$\mathscr{L}\varphi = f, \tag{3.14}$$

where $\varphi, f \in T(M, \boldsymbol{L}^*)$ it is clear that $f \perp \mathscr{H}$, as in (3.10), since $\mathscr{L}^* = \mathscr{L}$.

3.15 Proposition. *Suppose that for every $f \in \Gamma(M, \boldsymbol{L}^*)$ with $f \perp \mathscr{H}$ there exists a $\varphi \in \Gamma(M, \boldsymbol{L}^*)$ satisfying* (3.14), *then for every f which satisfies the necessary conditions given in* 3.13 *there exists a $u \in C^\infty(M)$ such that $\mathscr{D}u = f$.*

Proof. We have $f \perp \mathscr{H}$ and $\mathscr{D}f = 0$, furthermore

$$f = \mathscr{D}\mathscr{D}^*\varphi + \mathscr{D}^*\mathscr{D}\varphi.$$

We will show that $\mathscr{D}^*\mathscr{D}\varphi = 0$ and then setting $u = \mathscr{D}^*\varphi$ will complete the proof. Since $\mathscr{D}f = 0$ and $\mathscr{D}^2 = 0$ we conclude that

$$\mathscr{D}\mathscr{D}^*\mathscr{D}\varphi = 0.$$

Taking inner products into $\mathscr{D}\varphi$,

$$(\mathscr{D}\mathscr{D}^*\mathscr{D}\varphi, \mathscr{D}\varphi) = \|\mathscr{D}^*\mathscr{D}\varphi\|^2 = 0$$

and hence $\mathscr{D}^*\mathscr{D}\varphi = 0$ as required.

Thus the global problem of solving the over-determined system (1.12) on a compact manifold is reduced to solving the determined system (3.14). To study the operator $\mathscr{L}$ we introduce the hermitian form $Q:\Gamma(\boldsymbol{L}^*, M) \times \Gamma(\boldsymbol{L}^*, M) \to \boldsymbol{C}$ defined by

$$Q(\varphi, \psi) = (\mathscr{D}\varphi, \mathscr{D}\psi) + (\mathscr{D}^*\varphi, \mathscr{D}^*\psi) + (\varphi, \psi). \tag{3.16}$$

There is a general theorem which asserts that if Q is compact with respect to L_2 (i.e. if a sequence $\varphi_v \in \Gamma(\boldsymbol{L}^*, M)$ is bounded in the sense $Q(\varphi_v, \varphi_v) < C$ then it has a subsequence that converges in L_2), then the space $\mathscr{H}$ is finite dimensional and the system (3.14) has a unique solution in $\Gamma(\boldsymbol{L}^*, M)$ which is orthogonal to $\mathscr{H}$ (see [**16**]). An example of W. Sweeney (see [**31**]) shows that there are differential complexes for which the energy form is not compact and whose Laplacian, nevertheless, can be inverted. Here, however, we will discuss the problem of finding circumstances under which Q is compact.

4. **Hypoellipticity and the case of co-dimension one.** To show that Q is compact (and that $\mathscr{L}$ is hypoelliptic) it suffices to show that every point of M has a neighborhood U on which there are vector fields $Z_1, \ldots, Z_q$ such that the Lie algebra generated by these equals $\Gamma(CT, U)$ and that there exists $C > 0$ such that

$$\sum_{r,s} \|Z_r\varphi_s\|^2 \leqq CQ(\varphi, \varphi) \tag{4.1}$$

for all $\varphi \in \Gamma(\boldsymbol{L}^*, U)$ with compact support, here φ_s denote the components of φ relative to some fixed local basis of $\Gamma(\boldsymbol{L}^*, U)$. The special case where the Z's and single Lie brackets span all tangent vector fields was proved by the author (see [**14**]); the general case was proved by Hörmander (see [**10**], a proof based on pseudo-differential operators was found independently by Radkevitch [**27**] and the author [**17**]).

Let $L_1, \ldots, L_{m-p}$ be a local orthonormal basis of $\Gamma(\boldsymbol{L}_0, U)$, where U is a small neighborhood in M, and let $A_1, \ldots, A_p$ and $N_1, \ldots, N_{n+p-2m}$ be local orthogonal bases of $\Gamma(\boldsymbol{L} \cap \bar{\boldsymbol{L}}, U)$ and $\Gamma(N, U)$ respectively. Thus $L_1, \ldots, L_{m-p}, \bar{L}_1, \ldots, \bar{L}_{m-p}, A_1, \ldots, A_p, N_1, \ldots, N_{n+p-2m}$ is a local basis for the tangent vector fields on U. Let $\omega', \ldots, \omega^{m-p}, \bar{\omega}', \ldots, \bar{\omega}^{m-p}, \alpha^1, \ldots, \alpha^p, \eta', \ldots, \eta^{n+p-2m}$ be the corresponding local basis of the 1-forms. Then if $\varphi \in \Gamma(\boldsymbol{L}^*, U)$ we can write

$$\varphi = \sum_{j=1}^{m-p} \varphi_j \omega^j + \sum_{j=1}^{p} \varphi_{j+m-p} \alpha^j \tag{4.2}$$

and we have

$$\mathscr{D}u = \sum_{j=1}^{m-p} (L_j u)\omega^j + \sum_{j=1}^{p} (A_j u)\alpha^j. \tag{4.3}$$

Further a simple calculation shows that if

$$\mathscr{L}\varphi = \sum_{k=1}^{m-1} (\mathscr{L}\varphi)_k \omega^k + \sum_{k=1}^{p} (\mathscr{L}\varphi)_{k+m-p} \alpha^k$$

then

$$(4.4)\qquad \begin{aligned} (\mathscr{L}\varphi)_k = -\left(\sum_{j=1}^{m-p} \bar{L}_j L_j + \sum_{j=1}^{p} A_j^2\right)\varphi_k + \sum_{i=1}^{m-p} [\bar{L}_i, L_k]\varphi_i \\ + \text{ combinations of the } L, \bar{L} \text{ and } A, \\ \text{when } k = 1, \ldots, m-p, \end{aligned}$$

and

$$\begin{aligned} (\mathscr{L}\varphi)_k = -\left(\sum_{j=1}^{m-p} \bar{L}_j L_j + \sum_{j=1}^{p} A_j^2\right)\varphi_k \\ + \text{ combinations of the } L, \bar{L} \text{ and } A, \\ \text{when } k = m-p+1, \ldots, m. \end{aligned}$$

Note that the dimension of the fiber of $\boldsymbol{L} + \bar{\boldsymbol{L}}$ is $2m - p$ and that the dimension of the fiber of $\boldsymbol{N}$ equals the co-dimension of $\boldsymbol{L} + \bar{\boldsymbol{L}}$ equals $n + p - 2m$. Suppose that the co-dimension is one. We choose $N_1 = N$ such that $N = -\bar{N}$. Then we have

$$(4.5)\qquad [\bar{L}_i, L_k] = c_{ik}N + \text{ combinations of the } L, \bar{L} \text{ and } A.$$

In the case where the $\boldsymbol{L}$ is obtained on a submanifold of a complex manifold of real co-dimension one the (c_{ik}) is the Levi-form.

4.6 Theorem. *If the bundle $\boldsymbol{L} \subset \boldsymbol{C}T$ satisfies conditions* A, B *and* C *and if the fibers of $\boldsymbol{L} + \bar{\boldsymbol{L}}$ have co-dimension one, then the following are equivalent*: (a) *each point has a neighborhood U such that there exist $C > 0$ such that*

$$\sum_{j,k} \|L_j\varphi_k\|^2 + \sum_{j,k} \|\bar{L}_j\varphi_k\|^2 + \sum_{j,k} \|A_j\varphi_k\|^2 \leqq CQ(\varphi, \varphi)$$

for all $\varphi \in \Gamma(\boldsymbol{L}^*, U)$ with compact support; *and* (b) *$m \geqq 2$ and the matrix (c_{ik}) has either all eigenvalues of the same sign or has two pairs of eigenvalues of opposite sign.*

The proof of this theorem can be obtained by an approximate "uncoupling" of the above system. For a fixed point $P \in M$ we choose the orthogonal local basis $L_1, \ldots, L_{m-p}$ such that $c_{ik}(P) = \delta_{ik}\lambda_k$, where the (c_{ik}) are given by (4.5). Now we define the differential operators $P_k: C^\infty(U) \to C^\infty(U)$, $k = 1, \ldots, m - p$, by

$$P_k u = -\left(\sum_{j=1}^{m-p} \bar{L}_j L_j + \sum_{j=1}^{p} A_j^2\right) u + \lambda_k N u. \tag{4.7}$$

4.8 Theorem. *The following are equivalent:*

(a) *There exists $C > 0$ such that*

$$\operatorname{Re}(P_k u, u) + \|u\|^2 \geqq C\left(\sum_{j=1}^{m-p} \|L_j u\|^2 + \sum_{j=1}^{m-p} \|\bar{L}_j u\|^2 + \sum_{j=1}^{p} \|A_j u\|\right)^2 \tag{4.9}$$

for all $u \in C_0^\infty(U)$.

(b) *Either $\lambda_k \neq 0$ and there is another eigenvalue of the same sign or $\lambda_k = 0$ and there are two nonzero eigenvalues of opposite sign.*

It then follows that if U is sufficiently small and if (4.9) *holds for $k = 1, \ldots, m - p$ then* (4.7) *also holds.*

For the proof of this see [**15**].

The imaginary part of the first order terms in the operators P_k plays a vital role here, so that they do not fall under the theory developed by Hörmander and Radkevitch (see [**10**] and [**27**]) in which the subprincipal part is real. A similar phenomenon as that presented by the operators P_k is found in the work of Grushin (see [**6**]). He considers special types of equations in $\boldsymbol{R}^n$ with polynomial coefficients which are elliptic outside of a linear subspace. One of his examples is as follows:

$$Pu = \frac{\partial^2 u}{\partial x_n^2} + x_n^2 \sum_{j=1}^{n-1} \frac{\partial^2 u}{\partial x_j} + i\theta \frac{\partial u}{\partial x_1} \tag{4.10}$$

where $\theta \in \boldsymbol{C}$. In [**6**] the following is proved:

4.11 Proposition. *P is hypoelliptic if and only if either* $\operatorname{Im}\theta \neq 0$ *or* $\operatorname{Im}\theta = 0$ *and $|\theta| < 1$ when $n > 2$ or* $\operatorname{Im}\theta = 0$ *and θ does not equal an odd integer.*

Using the methods above, part of this proposition can be generalized as follows (see [**15**]).

4.12 Proposition. *If $X_1, \ldots, X_k$ are real vector fields defined on a neighborhood $U \subset \boldsymbol{R}^n$ and if the $\{X_j\}$ and $[X_i, X_j]$ span all the tangent vector fields then the operator*

$$Pu = -\sum_{j=1}^{k} X_j^2 u + i\theta[X_1, X_2]u + cu \tag{4.13}$$

where $\theta \in \boldsymbol{C}$ and $c \in C^\infty(U)$ is hypoelliptic whenever $\operatorname{Im}\theta \neq 0$ *and whenever* $\operatorname{Im}\theta = 0$ *and* $|\theta| < 1$. *In fact these conditions are necessary and sufficient for the estimate*

$$\text{(4.14)} \qquad \operatorname{Re}(Pu, u) + \|u\|^2 \geqq c \sum_{j=1}^{k} \|X_j u\|^2$$

for all $u \in C_0^\infty(U)$ *with support in a fixed compact subset of* U.

5. **The local problem.** The local problem is considerably more difficult than the global one (on compact manifolds). This can be seen already in the case of integrable almost complex structure where global existence is obtained from standard elliptic theory, but the local theorem depends on the Newlander-Nirenberg theorem. Now there are three distinct proofs of the Newlander-Nirenberg theorem, the original proof (see [**23**]) and the proofs given in [**13**] and [**22**]. The original proof depends heavily on the Cauchy formula so it seems more promising to generalize the latter two. To generalize the proofs of [**13**] one needs to understand the restriction of $\boldsymbol{L}$ to the boundary—this presents two difficulties: first that the higher co-dimension case is not yet understood, and second that the restriction will in general not satisfy condition B. There is some hope that these difficulties can be overcome, possibly by developing methods analogous to those in [**19**]. Malgrange's method of proof depends on the fact that solutions of analytic elliptic equations are analytic; it is possible that this is also true for the corresponding equations obtained from our problems. Recent results of Treves (see [**32**]) indicate that this may be so.

In conclusion we remark that very general overdetermined systems can be reduced, using the Spencer resolution (see [**30**]) to the ones discussed here (see also [**5**]).

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