

All in all, the subject is certainly an interesting one, and Donoghue's book is a beautifully written, excellent account of it. It is to be highly recommended to expert and beginner alike.

As for mistakes, there do not seem to be many. The consistent misspelling of Hans Bremermann's name is probably nothing but the final proof that Springer has become a naturalized American publisher.

ADAM KORÁNYI

*Homology in group theory*, by Urs Stambach, Lecture Notes in Mathematics, Volume 359, Springer-Verlag, New York, 1973, vii+183 pp., \$7.00

Cohomology theory is still viewed by many group theorists with a mixture of suspicion and indifference. There are, of course, good reasons for this. Homology theory began to invade group theory just before the Second World War, but it was not until the early fifties that the theory had been satisfactorily rebuilt as a completely algebraic tool. Group theorists then discovered that they had actually been practising homological algebra for years without knowing it: in Schur's theory of covering groups; in extension theory as developed by Schreier and Baer; and even in Burnside's transfer map. But homological algebra did not seem to be more than a satisfactory setting for work already done.

This situation has begun to change in the last fifteen years. The reasons perhaps can be grouped under three headings. First of all, a number of distinctly nontrivial new results have been established by homological methods. Samples: the Stallings-Swan theorem that a group  $G$  is free precisely if every extension by  $G$  with abelian kernel is split; the result of Gaschütz that every finite  $p$ -group has an outer automorphism of order  $p$ ; the very recent theorem of Bieri that a finitely presentable group, of cohomological dimension 2 and with a nontrivial centre, has a free commutator group.

Secondly, the outlook engendered by homological algebra can surely be held responsible for the spectacular development of integral representation theory, due to Swan and many others, as well as to important progress in modular representation theory such as Green's theory of sources and vertices.

Finally, the homological language has shown itself to be the natural one in which to express a good deal of the post-war work on generalised nilpotent groups.

Stambach's book is, above all, a contribution under this third heading. A good two-thirds of it deals with centrality properties of groups: the lower central series (Chapter 4), central extensions (Chapter 5) and localization of nilpotent groups (Chapter 6). Probably a more realistic (albeit more cumbersome!) title for the book would have been *Homological methods in the study of nilpotency properties of groups*.

The treatment of these topics is smooth, coherent and often elegant. It should help to convince group theorists of the efficacy of homological

methods. The author is also right to hope that by using techniques that have become standard equipment for a wide range of mathematicians, the whole of this area will become more accessible to everyone.

Chapters 1 and 2 are introductory. In the first, the group-theoretic preliminaries are dealt with: the notion of a variety, coproducts in varieties and various examples related to central series. The second chapter is a brief review of homological algebra. It is not an introduction for the novice, but can certainly be read by anyone knowing the elementary facts about the Ext and Tor functors. The author treats his book with Hilton (P. J. Hilton and U. Stambach, *A course in homological algebra*, Graduate Texts in Math., Springer) as the basic reference.

Chapter 3 introduces a generalization of the functors  $H^2(G, \#)$  and  $H_2(G, \#)$  from the variety of all groups to that of an arbitrary variety  $\mathfrak{B}$ . There is a slick definition: If  $F \rightarrow G$  is a  $\mathfrak{B}$ -free presentation (meaning that  $G$  is in  $\mathfrak{B}$  and  $F$  is  $\mathfrak{B}$ -free), then  $\hat{V}(G, \#) = \text{Ker}(H^2(G, \#) \rightarrow H^2(F, \#))$ . A dual definition for homology yields  $V(G, \#)$ . The functor  $\hat{V}(G, \#)$  classifies the extensions by  $G$  that lie in the variety  $\mathfrak{B}$  (p. 45). The whole of this chapter is a good introduction to the work of Leedham-Green, who has probably done more in this direction than anyone else.

The chapter on the lower central series (Chapter 4) is one of the best in the book. It contains an excellently organised account of the proof and consequences of the following result of Stallings and Stambach (pp. 64 and 77): Let  $f: K \rightarrow G$  be a homomorphism of groups in a variety  $\mathfrak{B}$  so that  $f$  induces an isomorphism on the abelian factor groups. If also  $f$  induces an epimorphism  $V(K) \rightarrow V(G)$  or  $V(G) \rightarrow V(G/G_i)$  is the zero map for all  $i \geq 2$  ( $G = G_1 \geq G_2 \geq \dots$  being the lower central series of  $G$ ), then  $f$  induces an isomorphism  $K/K_i \xrightarrow{\sim} G/G_i$  and a monomorphism  $K/K_\omega \rightarrow G/G_\omega$ . This result leads to theorems on subgroups of free groups in a variety, the structure of splitting groups (including Hall's results on nilpotent varieties), some of Baumslag's theorems on parafree groups and the "Huppert-Thompson-Tate theorem". The subject matter of the chapter is related to the theory of profinite and pronilpotent groups. (Tate's characterization of free pro- $p$ -groups can almost be viewed as another version of the Stallings-Stambach theorem!) It is a pity that no mention is made of this. Even so, this is a valuable chapter. But a warning: from p. 67 on,  $VG$  changes its meaning from being the verbal subgroup of  $G$  associated with  $\mathfrak{B}$  (p. 5) to  $V(G, \mathbf{Z})$  (p. 38).

Chapter 5 is devoted to central extensions. The unifying thread here is the 5-term exact homology sequence augmented by the extra term discovered by Ganea. If  $N$  is central in  $G$ , then multiplication  $N \times G \rightarrow G$  induces  $\mu: H_*(N) \otimes H_*(G) \rightarrow H_*(G)$  and  $\mu$  restricted to  $H_1(N) \otimes H_1(G)$  is the Ganea term  $N \otimes (G/G') \rightarrow H_2G$ . This can, of course, be defined directly by means of a free presentation of  $G$  (p. 104). The mapping  $\mu$  is shown to make  $H_*(N)$  into an associative, commutative, graded ring and  $H_*(G)$  into a graded  $H_*(N)$ -module. This material (§§1 and 4) is very pretty, but demands rather more sophistication on the part of the reader than the remainder of the chapter.

The "six-term sequence" is applied to giving estimates on the size of the Schur multiplier of finite groups as well as of finitely generated nilpotent groups (§9); and also, in the last section, to deriving D. Robinson's results related to the theorem of P. Hall that if  $N$  is a nilpotent normal subgroup of  $G$  and  $G/[N, N]$  is nilpotent, then  $G$  is nilpotent.

The chapter contains two further topics: a discussion of terminal groups based on a paper of L. Evens, Illinois J. Math **12** (1968); and the original theory of Schur approached by way of stem extensions and stem covers (in the sense of the reviewer: Springer Lecture Notes, No. 143, §9.9). Evens in his paper was concerned to develop a (co)homological process for constructing  $p$ -groups. Unfortunately this is never made explicit here. The uninitiated reader would find §8 rather unmotivated. In §3, stem extensions and stem covers are given a nice definition in terms of the map  $H_2(G/N) \rightarrow N$  appearing in the "six-term sequence": if this map is surjective, the extension is said to be a stem extension, and if it is an isomorphism, it is a stem cover. Thus  $G$  is a stem extension precisely when  $N \cong G'$  (p. 111). But the group-theoretic meaning of stem cover is explained only when the quotient group  $G/N$  is perfect (p. 124). Yet it is very easy to state generally: If  $G/N \cong F/R$ , where  $F$  is a free group and  $S$  is a complement to  $R \cap F'$  in  $R$  modulo  $[R, F]$ , then  $F/S$  is a stem cover. The categorical meaning of the various types of central extensions is never discussed, though this is, I believe, important for an understanding of their significance.

The final chapter (localization of nilpotent groups) is an account of Hilton's smooth and painless approach to the classical results of Mal'cev on "completions" of nilpotent groups. It also contains some new work of the author. The central concept here is that of an HPL-group  $G$ , where  $P$  is a given set of prime numbers: If the groups  $H_n(G)$  for all  $n \geq 1$  are  $\mathbf{Z}_P$ -modules ( $\mathbf{Z}_P$  is the localization of  $\mathbf{Z}$  at  $P$ ), then  $G$  is called an HPL-group. If  $G$  is nilpotent, then  $G$  is an HPL-group if, and only if,  $G$  has unique  $P'$ -roots; and this happens if, and only if, all factors of the lower (and upper) central series are  $\mathbf{Z}_P$ -modules. At the beginning of the chapter, the reader is presented with a long list of (for the most part) rather weighty papers dealing with other approaches to this theory. It is a pity that he could not have been led by the hand through some of this literature, to have had pointed out just where the material here fits in with earlier work.

All in all, this is an interesting book, though the style of the writing does make it a little difficult to decide for exactly what class of reader it is intended. Parts are quite elementary, but a good deal reads like a research monograph. The homologist may find that not enough explanatory pointers to the existing group theory are provided. The historical comments are often vague and no consistent method seems to have been adopted for attributing results. The group theorist, on the other hand, fares better. He might, perhaps, feel that the unrelenting use of exact sequences and tensor products is overdone: an occasional Hasse diagram can be more effective than a batch of exact sequences!

The mathematics concerns some interesting group theory presented from

an unfamiliar, but certainly valuable, point of view. The book should be in the hands of anyone working in nilpotent group theory (in the widest sense) and of anyone interested in seeing homological techniques put to work in group theory.

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*Aspects of constructibility*, by Keith J. Devlin, Lecture Notes in Math., Vol. 354, Springer-Verlag, New York, 1973, xii+240 pp., 22DM-

Gian Carlo Rota [1] named the years 1930–1965 as “The Golden Age of Set Theory.” Some of the results presented in this book call for extending the Golden Age beyond 1965. What made 1965 seem like a natural end to the Golden Age is, of course, P. J. Cohen’s proof (1963) of the independence of the axiom of choice and of the continuum hypothesis. What followed was a rich crop of independence proofs which showed that a long list of “classical” problems were not solved exactly because they cannot be decided on the basis of the presently accepted axiom systems for set theory. The fact that there are statements in set theory which are independent of the axiom system is no surprise to anybody aware of Gödel’s Incompleteness Theorem. The remarkable fact is that the problems which are now proven to be independent are more “natural” in the sense that they are raised by mathematical practice and not especially conceived so as to show independence. We do not claim that the problem of consistency of set theory is not natural for the student of foundations, but that for some reason the working mathematician is not bothered by it.

One direction (the consistency) of the independence of the continuum hypothesis was proved by Gödel in 1938 by introducing the constructible universe, which is a subclass or subcollection of the class of all sets (not necessarily a proper subclass). Gödel showed that the constructible universe is a model of all the axioms of set theory (including the axiom of choice). In fact, this class of all the constructible sets (usually denoted by  $L$ ) is the smallest such model including all the ordinals. Besides,  $L$  satisfies the generalized continuum hypothesis, i.e.  $2^{\aleph_\alpha} = \aleph_{\alpha+1}$  for every infinite cardinal  $\aleph_\alpha$ . Gödel also noted that some other problems are settled once we restrict ourselves to the universe of the constructible sets, i.e. we assume  $V=L$  where  $V$  is the class of all sets.

It will not be an exaggeration to suggest that after the introduction of  $L$  by Gödel, the major part of the study of  $L$  is due to one person, Ronald B. Jensen. Jensen’s results in the 60’s and 70’s show that the majority of the problems which were shown to be independent of the axioms of set theory are settled once we assume  $V=L$ . (That includes Souslin Problem, Kurepa Hypothesis, different partition problems, two cardinals problems in model theory, etc.) We do not claim, of course, that every problem in Set Theory is settled by  $V=L$ , Gödel’s Theorem forbids!, but in some ill defined sense, every “natural” problem seems to be decided in the constructible universe.