(3) (see p. 425) Electromagnetic interactions are weaker than weak interactions. In the category of misleading proofs, one example will suffice: The proof on p. 88 that a representation has a dense subspace of differentiable vectors fails to use an approximate identity.

Despite the errors, Želobenko's book is a positive contribution because it assembles so much useful information. The researcher in Lie groups will want to own it. The teacher or student in Lie groups will want to own the excellent book by Varadarajan instead.

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Analyse differentielle, by Valentin Poenaru, Lecture Notes in Mathematics, vol. 371, Springer-Verlag, New York, 1974, 228 pp., $7.70

Stable mappings and their singularities, by M. Golubitsky and V. Guillemin, Graduate Texts in Mathematics, vol. 14, Springer-Verlag, New York, 1973, x+209 pp., $9.50

Two names dominate the early study of singularities of smooth (C^∞) maps: those of René Thom and Hassler Whitney. The contents of these books essentially consist of expositions of their works and of the outstanding, more recent work of John Mather dealing with smooth stability.

The two books appear in different series published by Springer-Verlag, Golubitsky and Guillemin (G^2) in the series *Graduate texts in mathematics* and Poenaru (P) in the series *Lecture notes in mathematics*, and their styles reflect that fact. G^2 have written a careful, clear textbook, often improving on the existing expositions, invariably putting a personal stamp on the material discussed and supplying the reader with useful exercises (particularly those in later chapters). Their aim is "to present to first and second year graduate students a beautiful and relatively accessible field of mathematics—the theory of singularities of stable differentiable mappings." The value of P's book, on the other hand, is in his having gathered together material from a number of sources. His expositions remain quite close to those of his cited sources, but he has often included more detail. Gaps that appear in the frequently elliptical writing of research publications have been filled in. Some background material is outlined where needed but for the most part it seems to me that P is aiming his book at the reader who may lack experience but otherwise has all the prerequisites to read the original sources.

In the following paragraphs I will try to give some idea of the contents of the two books referring to their chapters as G^2-III or P-IV_i etc. ($i=1, 2$ referring to the semesters into which P is divided.)

Here manifolds X, Y, etc. will be smooth and *compact.* I will suppress ∞ as much as possible, so $C(X, Y)$ means the set of smooth (C^∞)-maps from X to Y, $C(X)$ means $C(X, \mathbf{R})$, $\Gamma(E)$ means the set of smooth sections of a bundle E. I will use subscripts to denote germs: $C_x(X, Y)$ is the set of germs at x of smooth maps from X to Y; $C_{x,y}(X, Y)$ are those germs at x with value y; $\Gamma_x(E)$ are the germs of sections at x in the base, etc. Given a bundle B with base X, B_x will mean the fibre at x; for the tangent bundle TX, T_xX is the fibre at x. Throughout $\dim X=n$ and $\dim Y=p$.

1. The definition of a kth order singularity of a smooth germ $f\in C_{0,0}(\mathbf{R}^n, \mathbf{R}^p)$ was given by Thom as the orbit in $J^k(n, p)$ of the k-jet of f at 0, $j^kf(0)$. Here $J^k(n, p)$ is just the real vector space of all p-tuples of polynomials in n-variables of degree $\leqq k$ and $j^kf(0)$ is the p-tuple of kth order Taylor polynomials at 0 of the component germs of f. The orbit referred to is the one produced by the action

$$(\mathrm{Diff}_0\,\mathbf{R}^n\times\mathrm{Diff}_0\,\mathbf{R}^p)\times C_{0,0}(\mathbf{R}^n, \mathbf{R}^p)\to C_{0,0}(\mathbf{R}^n, \mathbf{R}^p)$$

$$(h, k), f\to k\circ f\circ h^{-1}$$

where the group $\mathrm{Diff}_0\,\mathbf{R}^n$ is the group of germs of diffeomorphisms at 0 of $\mathbf{R}^n$ with itself leaving 0 fixed. Since the action thus induced on $J^k(n, p)$ depends only on the k-jets of the elements of $(\mathrm{Diff}_0\,\mathbf{R}^n\times\mathrm{Diff}_0\,\mathbf{R}^p)$, $(L^k(n)\times L^k(p))$, the orbits in $J^k(n, p)$ are those of the Lie group $(L^k(n)\times L^k(p))$.

In the obvious way, using coordinate systems, one can mimic the preceding for smooth maps $C(X, Y)$ and obtain smooth bundles $J^k(X, Y)$ over $X\times Y$, where the fibre over (x, y), $J^k(X, Y)_{(x,y)}$ is diffeomorphic to $J^k(n, p)$ (via a choice of coordinates at x and at y). Also $(\mathrm{Diff}\,X\times\mathrm{Diff}\,Y)$ operating on $C(X, Y)$ induces an action on $J^k(X, Y)$ as above, whose orbits globalize the notion of a kth order singularity. These orbits are subbundles of $J^k(X, Y)$. If $S\subseteq J^k(n, p)$ is any subset invariant under $L^k(n)\times L^k(p)$, denote by $S(X, Y)$ the subbundle of $J^k(X, Y)$ having fibre S (via any—hence all—choices of coordinates).

Given a map $f\in C(X, Y)$ define $j^kf\in C(X, J^k(X, Y))$ as follows: For each $x\in X$, $y=f(x)\in Y$, choose coordinate systems $\varphi\in C_{0,x}(\mathbf{R}^n, X)$ and $\psi\in C_{0,y}(\mathbf{R}^p, Y)$. These coordinate systems define the diffeomorphism $J^k(X, Y)_{(x,y)}\to J^k(n, p)$; let $j^kf(x)$ be the pre-image of $j^k(\psi^{-1}\circ f\circ\varphi)(0)$. For any subset $W\subseteq J^k(X, Y)$, let $W(f)=(j^kf)^{-1}(W)$. If $W=S(X, Y)$ write $S(f)$ for $W(f)$. A map $f\in C(X, Y)$ is said to display a singularity S at x if $x\in S(f)$.

The notion of stability appears in the subject immediately if one is interested in those properties of mappings which persist under small perturbations. To make sense out of the last phrase means defining a topology on $C(X, Y)$. The topology that turns out to be appropriate to this study is the

Whitney topology a basis for which is the set of all $M(U)$ for all open $U \subset J^k(X, Y)$ and all k, where $M(U)=\{f \in C(X, Y) | j^k f(X) \subseteq U\}$. (The *compact-open Whitney topology* is the union of the pull backs for all k of the compact open topology on $C(X, J^k(X, Y))$ via $j^k: C(X, Y) \to C(X, J^k(X, Y))$.) In all of the special cases studied by Whitney, the maps considered were restricted to have some stability in their partial derivative behavior—the singularities they displayed did not change when the maps were perturbed a little.

Challenged by Whitney to do it, Thom proved the *transversality theorem* which conceptually simplified and unified all the known local sufficient conditions for this kind of stability. The notion of transversality is a generalization of general position. If $g \in C(N, P)$ and V is a submanifold of P, g is transversal to V if at each $x \in g^{-1}(V)$, $Tg(T_x N)+T_{g(x)}V={}_{g(x)}P$, the subspaces $Tg(T_x N)$ and $T_{g(x)}V$ are in general position in $T_{g(x)}P$. The Thom transversality theorem states:

If W is a submanifold of $J^k(X, Y)$, then the set of $f \in C(X, Y)$ for which $j^k f$ is transversal to W is residual in $C(X, Y)$.

John Mather realized that a condition of transversality that would guarantee stability would have to take into account self-intersections as well as intersections of the source manifold with singular submanifolds of the jet bundle. To this end he extended the transversality theorem to include the right generalization of normal crossings by introducing multijets. Let $X^{(s)}$ be the subset of the cartesian product X^s of s-tuples of distinct points of X. Let $\alpha: J^k(X, Y) \to X$ be the source map, the α-image of a jet over (x, y) is x. Then $\alpha^s: (J^k(X, Y))^s \to X^s$ and define ${}_sJ^k(X, Y)=(\alpha^s)^{-1}(X^{(s)})$. Given a map $f \in C(X, Y)$, define ${}_sj^k f=(j^k f)^s | X^{(s)}$. The multijet transversality theorem is:

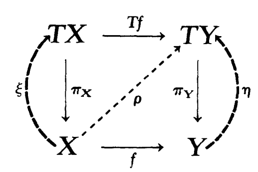
If W is a submanifold of ${}_sJ^k(X, Y)$, then the set of $f \in C(X, Y)$ such that ${}_sj^k f$ is transversal to W is residual in $C(X, Y)$.

The background for analysis on manifolds including the above mentioned transversality theorems is given in (G^2-I and II). Aside from the fact that a manifold structure is defined as a special kind of Γ-structure, these two chapters contain no surprises and comprise a carefully written rather complete introduction. They do appeal to outside references for a proof of the implicit function theorem and for integration of ordinary differential equations. However Sard's theorem that *the set of critical values of a smooth map has measure zero* is proved. The Whitney topology is discussed with care. They show how this topology differs from the compact open Whitney topology for a noncompact source and they prove that $C(X, Y)$ is a Baire space in the Whitney topology. The proof given of both versions of the transversality theorem is very close to Thom's original proof but is recast in a beautifully transparent form due to J. M. Boardman and John Mather.

Having the multijet transversality theorem available, G^2 prove, with little effort, the Whitney embedding theorem and the density theorem of Morse functions with distinct critical values. The notes of P contain none of this introductory material.

2. The definition of *stability* of $f \in C(X, Y)$ is that its orbit under the action

of (Diff X)×(Diff Y) be open in $C(X, Y)$. Since the subset of $C(X, Y)$ whose ${}_sj^k$-images are transversal to a submanifold of ${}_sJ^k(X, Y)$ is clearly invariant under the action of (Diff X)×(Diff Y), it is immediate from the transversality theorem that if f is stable, ${}_sj^kf$ must be transversal to the orbits of all ${}_sj^kf(x)$ for all $x \in X$ and all integers s and k. It is a strong form of the converse of this fact (for a *proper* f, the transversality of ${}_sj^kf$ for $k \geqq p = \dim Y$ and $s \geqq p+1$, to all its orbits implies stability of f) that is a part of stability theorem of John Mather. The proof of this theorem yields a number of conditions on f, each of which is equivalent to stability. The usual statement is, however, that *for a proper map* $f \in C(X, Y)$, *f stable is equivalent to f infinitesimally stable.* (The properness assumption is only relevant for X noncompact.) To define infinitesimal stability consider the diagram for the tangent map of f:



f is *infinitesimally stable* if given any $\rho \in C(X, TY)$ such that $\pi_Y \circ \rho = f$, there are vector fields ζ and η such that $Tf \circ \zeta + \eta \circ f = \rho$. There is the obvious vector space isomorphism between the space of such maps ρ over f, and the sections, $\Gamma(f^*TY)$. Composing with this isomorphism ($Tf\ \circ$) and ($\circ\ f$) gives linear maps f_* and f^* of sections $\Gamma(TX)$ and $\Gamma(TY)$ to $\Gamma(f^*TY)$. Thus infinitesimal stability of f is the surjectivity of

$$(f_* + f^*): \Gamma(TX) \otimes \Gamma(TY) \rightarrow \Gamma(f^*TY).$$

In their treatment of the stability theorem both G^2 and P restrict themselves to compact X and Y. After stating the theorem they motivate it as follows: If N and P are smooth compact manifolds, $C(N, P)$ is a (Fréchet) manifold whose tangent space at $g \in C(N, P)$ is just the linear space of smooth sections $\Gamma(g^*TP)$. (The fact that $C(N, P)$ is a Fréchet manifold is proved in G^2-III, as well as the fact that Diff N is an open subset of $C(N, N)$.) For $f \in C(X, Y)$, consider the smooth map

$$A: (\text{Diff } X) \times (\text{Diff } Y) \rightarrow C(X, Y): (h, k) \rightarrow k \circ f \circ h^{-1}.$$

The stability theorem is an immediate consequence of: A is open at the identity of the group (Diff X)×(Diff Y) iff the tangent map, TA, is surjective at that point, surjectivity of TA at the identity being precisely infinitesimal stability of f. Thus infinitesimal stability would imply stability if the implicit function theorem were true for smooth maps between Fréchet manifolds. However such an implicit function theorem is known to be false in general.

Having motivated the statement of the theorem, G^2-III continue with a number of examples of classes of stable maps. G^2 use the infinitesimal stability criterion to check the stability of submersions, Morse functions with distinct critical values and immersions with normal crossings. In fact they show that if $p = \dim Y > 2 \dim X = 2n$, then f is stable iff f is a 1:1 immersion and if $p = 2n$, then f is stable iff it is an immersion with normal crossings. The final class G^2

discuss in III is submersions with folds. These are maps $f \in C(X, Y)$ where $p = \dim Y \geqq \dim X = n$ and $(\operatorname{rank} f) \geqq p-1$ everywhere. Further by a transversality assumption, $S_1(f) = \{x \in X \mid (\operatorname{rank} f)(x) = p-1\}$ is a manifold and it is further assumed that $f|S_1(f)$ is an immersion. They prove that among these maps the only stable ones are those f such that $f|S_1(f)$ is an immersion with normal crossings. All of this last topic is a preview of their discussion of Thom-Boardman singularities in G^2-VI.

3. It is in the development of the deep Weierstrass-Malgrange-Mather division and preparation theorems and their applications to the proof of Mather's stability theorem that the books of G^2 and P have their largest overlap.

Only after Thom convinced him that it had to be true did Malgrange prove a division and preparation theorem for smooth functions **[M]**. Later Mather proved a strengthened form of the theorem, adapted for his use in proving the stability theorem.

A typical use of the generalized Malgrange preparation theorem is to conclude that a map between spaces of germs is surjective when the surjectivity is only known for the corresponding jet spaces; that is, under certain circumstances if one can solve functional equations up to a certain order, one can solve them.

In P-I_1 Poenaru reproduces Mather's original proof **[JM I]** and in P-II_1 follows Malgrange's treatment **[M]** of various forms of the preparation theorem including a useful equivalence between the preparation theorem and its formal power-series analog. Since they do not use the global division theorem, G^2-IV prove only the local version. The proof they give is due to Louis Nirenberg **[N]**. Both P and G^2 give the easier proof of the Weierstrass division theorem for holomorphic functions as an introduction to and motivation for the techniques that follow. They also apply the preparation theorem to prove Glaeser's: *Any symmetric germ in* $C_0(\mathbf{R}^n)$ *is the germ at* 0 *of a smooth function of the elementary symmetric functions.* Another application given by P is the Whitney theorem that *there is a dense set of smooth mappings between compact* 2*-manifolds which at each point of the source has one of three coordinate forms*: (*identity*, (x, y); *fold*, (x, y^2); *cusp*, $(x, xy-y^3)$). The analogous discussion of Whitney's theorem is used in G^2-VI as part of an introduction to the Thom-Boardman singularities.

4. An appealing geometric condition that G^2 in V prove equivalent to stability is stability under k-deformations. $f \in C(X, Y)$ is stable under k-deformations *if given any* $F \in C(X \times \mathbf{R}^k, Y)$ *with* $F_0 = f$, *there is a neighborhood of* 0, $U \subseteq \mathbf{R}^k$ *and maps* $G \in C(X \times \mathbf{R}^k, X)$ *and* $H \in C(Y \times \mathbf{R}^k, Y)$ *such that*:

$$G_0 = \mathrm{id}_X, \qquad H_0 = \mathrm{id}_Y,$$
$$F_t = H_t \circ f \circ G_t \quad \text{for all } t \in U.$$

(For $K \in C(A \times \mathbf{R}^k, B)$ write $K_t(a)$ for $K(a, t)$.)

It is not hard to see by integrating vector fields for one implication and using the chain rule for the other that this condition on F is equivalent to: *If* τ

is any germ at $X\times 0$ *of a vector field in* $X\times\mathbf{R}^k$ *with values in* $\pi^*T\mathbf{R}^k$, *there is a germ at* $X\times 0$ *of a vector field* X *in* $X\times\mathbf{R}^k$ *with values in* π_X^*TX, *and a germ at* $Y\times 0$ *of a vector field* Y *in* $Y\times\mathbf{R}^k$ *with values in* π_Y^*TY *such that*

$$F_*X+(F\times \mathrm{id}_{\mathbf{R}^k})^*Y=F_*\tau$$

where $\pi: X\times\mathbf{R}^k\to\mathbf{R}^k$, $\pi_X: X\times\mathbf{R}^k\to X$, and $\pi_Y: Y\times\mathbf{R}^k\to Y$ are projections.

Consider the map:

$$A_F=(F_*+(F\times\mathrm{id}_{\mathbf{R}^k})^*)\colon \Gamma_0(\pi_X^*TX)\oplus\Gamma_0(\pi_Y^*TY)\to\Gamma_0(F^*TY)$$

where the subscript 0 means that we are taking germs at $X\times 0$ or $Y\times 0$. If we restrict to the origin, $0\in\mathbf{R}^k$, this map becomes

$$A_f=(f_*+f^*)\colon \Gamma(TX)\oplus\Gamma(TY)\to\Gamma(f^*TY)$$

whose surjectivity is the condition for infinitesimal stability of f. Since it is obvious that any smooth section of f^*TY is the restriction to $X\times 0$ of a germ at $X\times 0$ of an $F_*\tau$ for an appropriately chosen F, we see that stability under k-deformations implies infinitesimal stability. The fact that the converse is true, that A_F, restricted to the vector fields on $X\times 0$ and $Y\times 0$, is surjective implies the surjectivity of A_F itself, is a consequence—and a typical one—of a global form of the Malgrange preparation theorem. (Actually G^2 use only a local form and piece things together.) Thus G^2 obtain the equivalence of infinitesimal stability with stability under k-deformations for all $k\geqq 1$. If we knew that stability under k-deformations was an open condition then we would know that it implied stability as well. By virtue of the above equivalence it is enough to show that infinitesimal stability is an open condition. This is essentially the route taken by G^2 in the first four sections of V to produce a new proof that infinitesimal stability implies stability. Their proof is a little simpler than Mather's and avoids the use of the global form of the division theorem. Following Mather they first show that the global condition of infinitesimal stability is equivalent to:

For any $y\in Y$ *and every subset* $S\subseteq f^{-1}(y)$ *where* $S=\{x_1,\cdots,x_k\}$ *with* $k\leqq 1+\dim Y$,

$$(*)\qquad f^*(\Gamma_S(TX))+f^*(\Gamma_y(TY))=\Gamma_S(f^*TY)$$

where the subscripts S, y *mean taking germs at* S, y.

The proof of this uses only the compactness of X (or properness of f). Now the preparation theorem is used to show that the above germ equation is satisfied if it is satisfied modulo functions that vanish on S to order $(p+1)$. Thus the global infinitesimal stability condition is equivalent to a condition on the $(p+1)$-jet of the mapping at $k\leqq(p+1)$ points of the source. Call this $(p+1)$-fold multijet of order $(p+1)$ condition, $(*_p)$. It is obvious that it is an open condition away from the generalized diagonal in X^k. G^2 get over this difficulty at the diagonal by a clever use of the equivalence of infinitesimal stability and stability under k-deformations for all k. In this way G^2 complete the proof that infinitesimal stability implies stability.

Poenaru's proof of this implication in II_2 follows the lines of Mather's proof

[**JM II**] closely, simplifying the exposition somewhat. He obtains, as did Mather, a stronger result than I have mentioned so far. Namely, if f is infinitesimally stable, and if g is close enough to f, then the diffeomorphisms h and k such that $g=h \circ f \circ k^{-1}$ can be chosen to depend continuously on g. In the definition of this strong form of stability on p. 125, Poenaru incorrectly requires that the diffeomorphisms h and k depend differentiably on g. The words "et differentiable" on p. 125, $3\downarrow$ should be deleted since such dependence is neither proved, referred to, nor used in any way. Since his proof makes no appeal to the local $(*_p)$-condition, he needs to use the global form of Mather's division theorem.

The proof that stability implies infinitesimal stability is very speedily handled by G^2 (in the last two sections of V) since they have the equivalence of condition $(*_p)$ to infinitesimal stability available. However both books follow Mather's argument [**JM V**] pretty closely. Stability of f implies transversality of ${}_rj^kf$ to all orbits of ${}_rJ^k(X, Y)$ for all r and k and then this transversality is translated into $(*_p)$. The treatment in P, following the original papers more closely, interposes the notion of contact classes (orbits in the jet space of a group which properly contains (Diff $X\times$Diff Y)). The computation of the tangent spaces to various jet spaces is carried out in P-III_2 and the end of the proof is given in P-IV_2.

5. The singularities in $J^1(X, Y)$ are particularly simple in that there are only finitely many orbits of (Diff X)$\times$(Diff Y): $S_0, S_1, \cdots, S_q$; $q=\min(n, p)$. Here S_r is the set of 1-jets of germs whose rank is $q-r$ (i.e. the rank of the tangent map at the source of the germ is $q-r$). G^2 prove that the S_r are submanifolds and give their codimensions. If one takes $f\in C(X, Y)$ with j^1f transversal to all S_r, $r=0, \cdots, q$, one obtains manifolds $S_r(f)\subseteq X$. The singularities of $f|S_r(f)$ are a next natural object of study and the immediate question is: Are there submanifolds $S_{r,s}\subseteq J^2(X, Y)$ such that if j^1f is transversal to all S_r, then

$$S_{r,s}(f)=\{x\in S_r(f) \mid \operatorname{rank}(f \mid S_r(f))(x)=\dim S_r(f)-s\}?$$

It is the general problem of this type that J. M. Boardman [**B**] solved.

Boardman not only established the existence of such manifolds S_I but computed their codimensions as well. G^2 in VI prove the existence of the subbundles $S_{r,s}$ and compute their codimensions. They do this very neatly, introducing for their proofs the intrinsic derivative of Ian Porteous for maps between vector bundles. They do not prove the theorem of Boardman for the analogous higher order singularities S_I nor unfortunately do they give their codimensions.

Now G^2 ask: Is a map stable if its jet extensions are transversal to all S_I simultaneously and which further satisfy the multijet transversality conditions that guarantee normal crossing behavior? The converse is obvious because the set of maps satisfying such a countable list of transversality conditions is residual and the set of stable maps is open. The question is answered *negatively* by showing that the set of stable maps between manifolds both of dimension n^2, $n\geqq 3$, is not dense. G^2 also give Mather's formulas for the pairs

(n, p) for which stable maps are dense in proper maps from n-manifolds to p-manifolds. If one relaxes the requirement of stability by saying that $f \in C(X, Y)$ is C^0-stable if any smooth g, sufficiently close to f, can be written as $k \circ f \circ h^{-1}$ for k and h merely homeomorphisms then *the C^0-stable maps are dense in $C(X, Y)$ for X compact.* This marvelous theorem of Thom-Mather is only mentioned in G^2 with reference to a (hopefully) forthcoming book of John Mather.

6. Since global stability of a map is equivalent to a condition at $(p+1)$-tuples of points it makes sense to ask if there is a germ form of the stability theorem. The germ form of infinitesimal stability is immediate—just interpret all maps as germs. To get a notion of a stable germ requires more care. For suppose $f \in C(X, Y)$ is stable, then surely the germs f_x for all $x \in X$ should be called stable. But if $g \in C(X, Y)$ is close to f and if $g = h \circ f \circ k^{-1}$, then the germ $f_x \in C_{x,y}(X, Y)$ is equivalent to $g_{x^1} \in C_{x^1,y^1}(X, Y)$ where $x^1 = k(x)$ and $y^1 = h(x)$. So the definition of stable germs has to allow such shifts in source and target of equivalent germs. The appropriate definitions and the germ stability theorem are given at the beginning of P-III$_2$.

Using the fact that germ infinitesimal stability depends only on the $(p+1)$-jet of the germ, one could hope to be able to classify all stable germs by constructing some invariant of the $(p+1)$-jet. Mather did exactly that in [**JM IV**]. Let $f \in C_{x,y}(X, Y)$ and define

$$Q(f) = C_x(X)/f^* m_y C_x(X) \quad \text{and} \quad Q_k(f) = C_x(X)/f^* m_y C_x(X) + m_x^{k+1} C_x(X)$$

where m_x (resp. m_y) is the maximal ideal of all germs vanishing at x (resp. y). The Mather theorem referred to above is that *two stable germs are equivalent (via germs in* (Diff $X \times$Diff Y)) *iff their Q_{p+1}'s are isomorphic.* This theorem is stated in both books without proof. What is proved is that *two k-jets of germs are contact equivalent iff their Q_k's are isomorphic.* Here two germs in $C_{x,y}(X, Y)$ are *contact equivalent* if there is a diffeomorphism germ belonging to a subgroup $\mathcal{K} \subset \mathrm{Diff}_{x \times y}(X \times Y)$ which takes the graph of one germ to that of the other. The group

$$\mathcal{K} = \{(\varphi, \psi) \in \mathrm{Diff}_{x\times y}(X \times Y) \mid (\bar{x}, \bar{y}) \to (\varphi(\bar{x}), \psi(\bar{x}, \bar{y})) \text{ and } \psi(\bar{x}, y) = y\}$$

is called the contact group. Putting these two results together tells us that a stable germ is determined up to equivalence by its $(p+1)$-jet.

The general question of when a k-jet of a germ determines the "equivalence" class of the germ is taken up in P-III$_2$. The various notions of "equivalence" considered are those induced by the action on $C_{x,y}(X, Y)$ of the group $\mathcal{R} = \mathrm{Diff}_x X$, $\mathcal{L} = \mathrm{Diff}_y Y$, $\mathcal{A} = \mathcal{R} \times \mathcal{L}$, $\mathcal{K}$ just defined above and $\mathcal{C} = \{(\varphi, \psi) \in \mathcal{K} \mid \varphi = \mathrm{id}_X\}$. Thus if G is any one of the above groups one defines $f \in C_{x,y}(X, Y)$ is *k G-determined* if $g \in C_{x,y}(X, Y)$ and $j^k g = j^k f$ imply g is G-equivalent to f. Or, *f is finitely G-determined* if it is k G-determined for some $k < \infty$. The theorem of Mather [**JM III**] that deals with this question is:

$f \in C_{x,y}(X, Y)$ is finitely G-determined iff the codimension of the G-orbit of f in $C_{x,y}(X, Y)$ is finite.

The interpretation of the finite codimension statement is: Let the action of G on f be denoted by $A: G \to C_{x,y}(X, Y): g \to g(f)$. If we consider the tangent map at the identity $e \in G$, we have $TA(T_eG) \subseteq T_f(C_{x,y}(X, Y)) = \Gamma_x(f^*TY)$. The finiteness required by the theorem is that of the codimension of $TA(T_eG)$ in $\Gamma_x(f^*TY)$. In case $G = \mathscr{A}$, and f is an infinitesimally stable germ, this codimension is zero. In this case the theorem says that stable germs are finitely $\mathscr{A}$-determined.

G^2 in VII obtain other types of results using Q. For example if $f \in C(X, Y)$ and $\dim_{\mathbf{R}} Q(f_x) = r < \infty$, then the map f near x is no more than r to one. They also give a weak global theorem: *A map each of whose germs has a finite dimensional Q has discrete fibres.*

Given any local ring $\mathscr{P}$, G^2 define the contact class $S_{\mathscr{P}} \subseteq J^k(X, Y)$ as the set of all k-jets of mapping germs f such that $Q_k(f) \cong \mathscr{P}$. By the above-mentioned theorem, this is a subbundle of $J^k(X, Y)$ whose fibre at (x, y) is a K-orbit. They prove that such classes are immersed submanifolds, thus transversality to $S_{\mathscr{P}}$ makes sense. They begin their study of such classes with the case $\mathscr{P} = \mathbf{R}[t]/(t^{k+1})$, calling these contact classes S_{1_k} the *Morin singularities.* They show that if j^kf is transversal to S_{1_k} for all k, then $S_{1_k}(f) = S_{1,1,\cdots,1}(f)$ where $S_{1,\cdots,1}$ is the Boardman singularity with k ones in the subscript. Using the preparation theorem they obtain normal forms for transversal S_{1_k}-points between manifolds of the same dimension. (These are the only higher order singularities that G^2 consider.) They derive the normal forms at transversal S_1-points for mapping of n into $(2n-1)$-manifolds and finally for all transversal S_2-points between equal dimensional manifolds whose Q-ring is a real 4-dimensional vector space. (This is the lowest possible dimension for Q of germs of maps between equal dimensional manifolds at transversal S_2-points.) G^2 finish their book with a characterization of stable maps and germs between equal dimensional manifolds both of dimension $\leqq 4$.

7. In the first part of Poenaru's book are found the division and preparation theorems in I_1 and II_1 (which I duscussed in §3) as well as Whitney's extension theorem in III_1, Łojasiewicz's theorem on regularly situated sets in IV_1 and Whitney's spectral synthesis theorem in V_1. The exposition of this sequence of topics is essentially that found in Malgrange's book [**M**], but with many details supplied which were omitted from Malgrange's terse treatment.

Similarly in I_2, P has included an expanded version of a short paper of Atiyah [**A**], which gives an application of Hironaka's resolution of singularities theorem to the problem of division of distributions. The material in this chapter is related to P-V_1, however it is more likely that its inclusion reflects P's enthusiasm for its contents rather than its compelling connection with the preceding chapters.

8. Aside from a number of minor misprints in P, the most serious one has already been mentioned: the required deletion on p. 125, $3\downarrow$ of "et differentiable." Also the "Remarques (heuristique)" on p. 132, 133 should be corrected as follows: $F \in C^{0,\infty}(Z \times I, C^\infty(X, Y))$ should be defined by $F(g) = F_g$ for $g \in Z$; on p. 132, $1\uparrow$ insert "$=A(g)$" before "definit"; on p. 133, $1\downarrow$ write

"F" instead of "F_g." Finally the exponents in Lemma 8, p. 190 need some fixing up.

Most of the errors in G^2 should not bother the reader. A few that might are: p. 96, 11 ↑ delete "t^k+"; p. 136 in the diagram, the target of the λ arrow should be $T_\sigma J^k(X, Y)$ instead of $T_\sigma D_\sigma$ which it contains; p. 156, Theorem 5.1 should begin: "For every sequence of integers $r_1+\max(0, \dim X-\dim Y)\geqq r_2\geqq\cdots$"; p. 171, 11 ↑ $\phi(x, y)=(x, H(x)y)$ and 5 ↑ "$\to$ graph f" should read "$\to(X, \{q\})$".

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The $P(\Phi)_2$ euclidean (quantum) field theory, by Barry Simon, Princeton University Press, Princeton, New Jersey, 1974, xx+392 pp., $20.00

During the last ten years, the mathematical development of quantum field theory has proceeded rapidly and in a direction that should be of interest to mathematicians working in functional analysis and stochastic processes. The presently most active branch of this area of mathematical physics, called constructive field theory, has succeeded by a delightful melange of analytical techniques in constructing a distinctly nontrivial generalized random process which is not Gaussian and has physical interest. The book under review presents a detailed account of this work and is substantially lectures given by the author at the ETH in Zurich during the Spring of 1973.