"F" instead of "F_g." Finally the exponents in Lemma 8, p. 190 need some fixing up.

Most of the errors in G^2 should not bother the reader. A few that might are: p. 96, 11 ↑ delete "t^k+"; p. 136 in the diagram, the target of the λ arrow should be $T_\sigma J^k(X, Y)$ instead of $T_\sigma D_\sigma$ which it contains; p. 156, Theorem 5.1 should begin: "For every sequence of integers $r_1+\max(0, \dim X-\dim Y)\geqq r_2\geqq\cdots$"; p. 171, 11 ↑ $\phi(x, y)=(x, H(x)y)$ and 5 ↑ "$\rightarrow$ graph f" should read "$\rightarrow(X, \{q\})$".

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The $P(\Phi)_2$ euclidean (quantum) field theory, by Barry Simon, Princeton University Press, Princeton, New Jersey, 1974, xx+392 pp., $20.00

During the last ten years, the mathematical development of quantum field theory has proceeded rapidly and in a direction that should be of interest to mathematicians working in functional analysis and stochastic processes. The presently most active branch of this area of mathematical physics, called constructive field theory, has succeeded by a delightful melange of analytical techniques in constructing a distinctly nontrivial generalized random process which is not Gaussian and has physical interest. The book under review presents a detailed account of this work and is substantially lectures given by the author at the ETH in Zurich during the Spring of 1973.

Quantum fields are operator-valued distributions introduced by theoretical physicists over the period 1930–1950 to be a basis for a dynamical theory of interactions between the fundamental microscopic particles occurring in nature. In addition to linear requirements of causality and special relativity, quantum fields must satisfy formal nonlinear partial differential equations which are not a priori well defined. It is a fundamental task of the mathematical theory to rectify this drawback in a manner that is consistent with physical requirements (the so-called renormalization program) and investigate the properties of the well-posed theory. A substantial first step along this path was the clarification of the mathematical framework by Wightman (1956)[1] and the subsequent penetrating study of its linear aspects. Even though the mathematically rigorous theory was able to contribute substantial knowledge and technique for use in theoretical physics, a valid nontrivial example was lacking. This would have settled questions concerning the internal consistency of this framework as well as suggesting avenues for further development. The work in this period 1955–1965 is well accounted in the monographs by Streater and Wightman (1964) and Jost (1965).

After this "classical" period, the search for examples or models was stimulated by Wightman's suggestion (1967) that the difficulties inherent in the dynamical models suggested by physicists might be ameliorated by their study in space and time dimensions lower than four. In this way a hierarchy of models presents itself in which the degree of singularity increases with the space dimension. This hierarchy is denoted according to the interaction contribution appearing in the formal Hamiltonian operator for the physical theory as $P(\Phi)_2$, Yukawa$_2$, $(\Phi^4)_3$, Yukawa$_3$, $(\Phi^4)_4$, Yukawa$_4$. The subscript stands for the dimension of space and time used in the related formalism and $P(x)$ denotes a semibounded polynomial. The use of the Hamiltonian operator as generator for a unitary group giving the evolution of the quantum field in time away from its noninteracting (free field) values circumvents some of the aforementioned difficulties. The issue is now selfadjointness and semiboundedness for the Hamiltonian. This task was first surmounted by Glimm in 1967 for the Yukawa$_2$ model and without the semiboundedness for $(\Phi^4)_3$ in 1968. As part of their extensive collaboration, Glimm and Jaffe refined and extended these results for Yukawa$_2$ (1970) proving semiboundedness for $(\Phi^4)_3$ by euclidean techniques in 1973. While these models are presently undergoing rapid development let us take up the history of $P(\Phi)_2$. This is the least singular model in the hierarchy and provides a complete example of Wightman's earlier framework. The semiboundedness for $(\Phi^4)_2$ was established by Nelson (1966) by probabilistic (euclidean) techniques which foreshadowed the future analysis of this model. Glimm (1968) extended this result to the more general $P(\Phi)_2$ case. At this point Glimm and Jaffe began and carried out a systematic study of this model from 1968 to 1974, culminating in a verification of spectral properties for the Hamiltonian

[1] Dates are given approximately to emphasize the chronological development of the subject and refer to the bibliography in the book under review.

with a $a\Phi^4+b\Phi^2-\mu\Phi$, $\mu\neq 0$, polynomial by Glimm, Jaffe and Spencer (1974). In Simon's book the appropriate references in the bibliography are **[62]**–**[74]** and the discussion in Chapter 10.

In the course of its development, probabilistic or euclidean techniques appeared in the study of the $P(\Phi)_2$ model with increasing frequency until they finally won the day becoming the principal setting for both the formulation of the mathematical theory as well as its technical apparatus. It is at this point that Simon's book begins the saga for $P(\Phi)_2$. Within quantum field theory this point of view may be traced to Schwinger (1958) and for $(\Phi^4)_2$ to Symanzik (1964) and Nelson (1966). The suggestion that euclidean methods might provide a complete basis for quantum field theory rather than a convenient technical tool is one of the theses advanced in the book under review. For the development of the overall picture, the reader could do no better than the introduction to this book.

It has been known for a long time that scalar free fields may be described as Gaussian generalized random processes. These are discussed in considerable detail in Chapter 1 along with various notions peculiar to quantum field theory such as Wick products and second quantization. The connection between the physicist's Fock space and the simultaneous diagonalization of time zero free fields realized as multiplication operators on $L_2(M, \mu)$ of a probability space accounts for the first half of this chapter. The second half is concerned with results about doubly Markovian operators (linear, bounded, positive operators T on $L_2(M, \mu)$ with $T1=T^*1=1$) and hypercontractive bounds on second quantized operators (Nelson **[131]**, **[135]**). For example, any doubly Markovian operator is a contraction on $L_p(M, \mu)$ and if A is a contraction between two real Hilbert spaces, the bound $\|A\|\leqq(p-1)^{1/2}(q-1)^{-1/2}$, $1<p\leqq q<\infty$, implies the second quantized operator $\Gamma(A)$ is a contraction from L_p to L_q, these spaces taken over the measure space for the underlying Gaussian process indexed by the two Hilbert spaces respectively. The bound on A is best possible for the result. These ideas form the mathematical basis for analyzing many operators appearing in the study of $P(\Phi)_2$.

Chapter 2 returns to the mainstream of the Wightman framework with a review and then an account of the connection between quantum fields and the euclidean ideas of Schwinger. A complete axiomatic equivalence was given by Osterwalder and Schrader (1973) by exploiting analytic continuation from real time to imaginary time in the linear Wightman program, thereby heuristically transforming a hyperbolic problem to an elliptic one. Though the author does not work through all details, several simplifications and a correction to the original publication may be found. This equivalence theorem between the Wightman distributions and euclidean Schwinger functions is an important addition to axiomatic quantum field theory. From this work one learns that the Schwinger functions are not uniquely defined at all values of time and space by the relativistic theory but rather must be regularized as distributions on the whole region. The remainder of this chapter and the next illustrates these ideas with free fields, particularly basic tools such as the Feynman-Kac-Nelson formula and Nelson's Markov property which are used in Chapter 5 to study the $P(\Phi)_2$ model. Probabilists will be interested in the proof of the Markov property and related aspects.

In Chapter 4 the transition from the relativistic viewpoint to the euclidean one is completed with a presentation of Nelson's euclidean theory over the Sobolev space $\mathcal{H}_{-1}$. Reconstruction of related quantum fields defined at sharp-time is achieved by means of the Osterwalder-Schrader axioms rather than verifying the Wightman axioms directly as originally done. Work by the author (1973) giving conditions within the relativistic framework which lead to a Nelson theory and the interdependence of the various axioms appears at an appropriate time. Recent work on this connection has centered on constructing Hahn-Banach extensions of the Osterwalder-Schrader Schwinger functions satisfying various positivity requirements. Though only begun, the thrust of this work is to see if a modification of Nelson's axioms exists which establishes a complete equivalence between euclidean and relativistic theories.

Chapter 5 begins the study of $P(\Phi)_2$ within the euclidean framework. The Hamiltonian with interaction term localized in space (space cut-off) is defined and the key results on selfadjointness and semiboundedness derived, together with bounds on its lowest eigenvalue (ground state energy) as a function of the cut-off in a form due to Nelson, Glimm and Segal. Uniqueness of the lowest eigenstate (ground state) is also true. A variety of important technical estimates appear along with the Schwinger functions for finite regions. In Chapter 6 the study of the ground state energy is continued, recovering and refining many of the basic estimates obtained by Glimm and Jaffe. Euclidean methods allow considerable simplification in their derivation due to an incisive insight by Nelson on the euclidean symmetry of space and time. The emphasis upon a priori estimates such as the behavior of the ground state energy as the space cut-off approaches a function identically one on the line stems from the control these estimates give when taking the infinite volume limit. Chapter 7 covers much the same ground as the previous chapter but with a subtle change in boundary conditions. Throughout the analysis of $P(\Phi)_2$ an important role is played by the kernel of $(-\Delta+m^2)^{-1}$ in the sense of distributions. For an open set $\Lambda\subset\mathbf{R}^2$, Dirichlet boundary conditions are obtained from the kernel of $(-\Delta_\Lambda+m^2)^{-1}$ where Δ_Λ is the Friedrichs' extension of Δ. Half-Dirichlet boundary conditions refer to a finite region measure constructed from the Gaussian measure for the free field with Dirichlet boundary conditions and the region being open, normal and satisfying L_p-type growth conditions near the boundary. These boundary conditions allow upper and lower bounds for the vacuum energy.

The last two main chapters, 8 and 9, discuss the main impact of the euclidean approach to the $P(\Phi)_2$ model due to Guerra, Rosen and Simon and Nelson (1973–1974). This model allows a discrete approximation by a lattice in $\mathbf{R}^2$ and very close analogies with the statistical mechanics of lattice systems. Chapter 8 brings to bear monotonicity techniques from statistical mechanics to show the existence of limits of half-Dirichlet finite region Schwinger