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IDEALS AND POWERS OF CARDINALS

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We obtain results concerning the behaviour of the function 2^{ω_α} ($\alpha \in \mathrm{On}$) under the assumption of the existence of certain kind of ideals. These results complement those of Ulam [7], Tarski [6] and Solovay [4] and [5]. In particular, it follows that if 2^ω is real-valued measurable, then $2^\nu = 2^\omega$ for all infinite $\nu < 2^\omega$.

We assume some familiarity with [4] and [5]. $\alpha, \beta, \gamma, \delta, \eta, \xi, \rho$ ($\kappa, \lambda, \nu, \tau$) denote ordinals (inf. cardinals). f, g, h denote functions; F denotes families of functions or sets. We use the Erdös-Hajnal notation $[S]^\nu$, $[S]^{<\nu}$, etc. (see [2]). F is λ-almost disjoint (λ-a.d.) if $|X \cap Y| < \lambda$ whenever $X, Y \in F$ and $X \neq Y$.

DEFINITION 1. κ is λ-real-supercompact (abbrev. λ-r.s.c.) if there is a real-valued κ-compl. measure μ defined on $P([\lambda]^{<\kappa})$ such that

(i) $\mu([\lambda]^{<\kappa}) = 1$;

(ii) for every $\alpha \in \lambda$, $\mu(\{x: \alpha \notin x\}) = 0$;

(iii) if $\mu(X) > 0$ and $f: X \longrightarrow \lambda$ is such that $f(x) \in x$ for all $x \in X$, then there is $Y \subseteq X$ such that $\mu(Y) > 0$ and f is constant on Y.

κ is r.s.c. if κ is λ-r.s.c. for all regular $\lambda \geqslant \kappa$. We define "κ is ω_1-saturatedly supercompact" (abbrev. ω_1-s.s.c.) by replacing μ by an ideal I in the obvious way.

One can show by the methods of [3] and [4] that if it is consistent that a s.c. cardinal exists, then it is consistent that 2^ω is r.s.c.

DEFINITION 2. $R_2(\kappa_0, \kappa_1)$ holds if for every partition $[\kappa_1]^2 = \bigcup\{K_\xi: \xi \in \lambda\}$, where $\omega < \lambda < \kappa_0$, there exists an $X \subseteq \kappa_1$ and $M \subseteq \lambda$ such that $|X| = \kappa_0$, $|M| < \lambda$, and $[X]^2 \subseteq \bigcup\{K_\xi: \xi \in M\}$.

THEOREM 1. *Let* $\lambda, \nu < \kappa$, $\omega < \mathrm{cf}(\lambda)$ *and* $F \subseteq [\nu]^{\geqslant\lambda}$ *be* λ*-a.d. If* $R_2(\kappa, \kappa)$ *holds and* $\mathrm{cf}(\kappa) > \omega$, *then* $|F| < \kappa$. *If* $R_2(\kappa, \kappa_1)$ *holds and* κ_1 *is regular, then* $|F| < \kappa_1$.

THEOREM 2. *Set* $2^\omega = \kappa$ *and suppose that* κ *carries a* κ*-compl.* ω_1*-sat. nontrivial ideal. Then*

(a) *for all* $\nu < \kappa$, $2^\nu = \kappa$;

(b) *if* $I \subseteq P(\kappa)$ *is* ω_1*-compl.,* ω_1*-sat. and* $[\kappa]^{<\kappa} \subseteq I$, *then* $|P(\kappa)/I| = 2^\kappa$;

(c) *if* $\nu < \kappa$ *and* $\mathrm{cf}(\nu) > \omega$, *then there is a family* $F \subseteq {}^\nu\nu$ *such that* $|F| < \kappa$ *and each* $g \in {}^\nu\nu$ *is dominated everywhere by some* $f \in F$;

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(d) *if* λ, $\nu < \kappa$, $\omega < \mathrm{cf}(\lambda)$ *and* $F \subseteq [\nu]^{\geqslant\lambda}$ *is* λ*-a.d., then* $|F| < \kappa$.

THEOREM 3. *Suppose that* $2^\omega = \kappa$ *is* ω_1*-s.s.c. Then*

(a) $\lambda^{\mathfrak{G}} = \lambda$ *for all regular* $\lambda > \kappa$;

(b) $2^\nu = \nu^+$ *for all singular strong limit* $\nu > \kappa$;

(c) *if* $I \subseteq P(\kappa)$ *is* ω_1*-compl.,* ω_1*-sat.,* $[\kappa]^{<\kappa} \subseteq I$ *and* $P(\kappa)/I$ *can be generated (by infinitary Boolean operations) from* λ *elements, then either* $2^\kappa = \lambda$, *or* $2^\kappa = \lambda^+$ *and* $\mathrm{cf}(\lambda) = \omega$;

(d) *if* $\lambda \geqslant \kappa$, *then* $\Box_\lambda$ *is false (see* [5] *for the statement of* $\Box_\lambda$).

Solovay [4, Lemma 14, p. 406] proved that $R_2(\kappa, \kappa)$ holds if κ carries a κ-compl. ω_1-sat. nontrivial ideal. The proof of Theorem 2(a) uses this result, Theorem 1, and Tarski's "almost disjoint sets" construction. It proceeds by induction on $\nu < 2^\omega$.

Theorem 2(b) strengthens a result of Kunen who showed that $|P(\kappa)/I| \geqslant \kappa^+$. To prove this, he used the fact that in the Boolean-valued universe $V^{P(\kappa)/I}$, $|P(\omega)| \geqslant \kappa^+$. Theorem 2(a) enables us to show that in $V^{P(\kappa)/I}$, $|P(\omega)| = 2^\kappa$.

To prove Theorem 2(c), we again use a method of Kunen who showed that the corresponding result holds for ${}^\omega\omega$ if 2^ω is r.v.m. This is made possible by Theorem 2(a). The method involves considering Solovay's Boolean ultrapower V^κ/I.

The proof of Theorem 3 involves ideas of [5, §§3 and 4] and an additional unpublished result of Solovay.

LEMMA 1 (SOLOVAY, UNPUBLISHED). *For every regular* $\lambda > \omega$ *there exists an* ω*-ary Jónsson algebra* $\langle \lambda, f \rangle$ *such that for every* $X \subseteq \lambda$, $|\mathrm{rng}(f \upharpoonright [X]^\omega)| \leqslant |X|$.

LEMMA 2. *Let* $\lambda \geqslant \kappa$ *be regular and* μ *be a measure as in Definition* 1.

(a) *If* $X \subseteq [\lambda]^{<\kappa}$ *and* $\mu(X) = 1$, *then* $|X| = \lambda^{\mathfrak{G}}$.

(b) *Let* g: $[\lambda]^{<\kappa} \longrightarrow \lambda$ *be defined by* $g(x) = \sup(x)$. *Then there is* $X \subseteq [\lambda]^{<\kappa}$ *such that* $\mu(X) = 1$ *and* $g \upharpoonright X$ *is one-to-one.*

The proof of Lemma 2(a) uses Theorem 2(a). Lemma 2(b) is analogous to Theorem 2 of [5]. The proof of Lemma 2(b) uses Lemma 1 where Solovay's proof of his Theorem 2 used an older result of [1]. Some modifications are required and this holds for the proof of Theorem 3(d) as well. Theorem 3(a) follows from Lemma 2 and implies Theorem 3(b). Theorem 3(c) follows from Theorem 2(b) and Theorem 3(a).

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