Elementary induction on abstract structures, by Yiannis Nicholas Moschovakis, Studies in Logic and the Foundations of Mathematics, vol. 77, North-Holland, Amsterdam; American Elsevier, London, New York, x+218 pp., $17.75.

This review is divided into three parts. In the first section we define, by means of examples, the objects studied in the book under review. The second section discusses the book itself. The third section is more technical. It discusses a way that the second-order assumption of "acceptability" imposed on structures in Chapter 5 can be weakened to make the theory developed in Chapters 5–8 relevant to ordinary first-order model theory.

Inductive definitions. The way to tell a logician from his mathematical colleague is by his attention to the language of mathematics. The logician takes as a fundamental tenet that light can be shed on mathematical problems by simply paying attention to, and then analyzing, the language in which mathematics is formulated and carried out. This book presents a detailed analysis of one part of the language of mathematics, namely inductive (to be precise, first-order positive inductive) definitions on a fixed structure $\mathfrak{A}$.

An *inductive definition* can be viewed as a monotone operator Γ on sets, monotone in the sense that $X\subseteq Y$ implies $\Gamma(X)\subseteq\Gamma(Y)$. It has associated with it certain *stages* $I_\Gamma^0\subseteq I_\Gamma^1\subseteq \cdots \subseteq I_\Gamma^\alpha \cdots$, a smallest *fixed point* I_Γ, and a *closure ordinal* $\|\Gamma\|$, equal to the least ordinal number β such that $I_\Gamma = \bigcup_{\alpha<\beta} I_\Gamma^\alpha$.

A. ERDÖS NUMBERS. We begin with a slightly frivolous example which shows that inductive definitions arise in real life. Let M be the set of mathematicians with a distinguished element $e\in M$. For $X\subseteq M$, let $\Gamma(X)$ be the set of those mathematicians who have published a joint paper p and one of the authors of p is in X. Let $I^0=\{e\}$, $I^{n+1}=\Gamma(I^n)$ and let I_Γ be the union of the various I^n. If e is properly chosen, then Γ is an inductive definition of the set of mathematicians that have Erdös numbers,[1] and I^n is the set of mathematicians with Erdös number $\leqq n$. Notice that I_Γ is a fixed point of Γ, $\Gamma(I_\Gamma)=I_\Gamma$, even if there are an infinite number of authors of some paper. Thus the closure ordinal is at most ω, the first infinite ordinal.

B. A MATE FOR WHITE IN α MOVES. Consider some two-person game like chess. Let W be the set of positions from which white has a winning strategy. We can give a more informative inductive definition of this set as follows. For any set X of positions, let $\Gamma(X)$ be the set of all positions such that, for any move of black, white has a response putting him in X. Let I^0 be the set of positions which are mates for white and let I^{n+1} be $\Gamma(I^n)$. Thus I^n is the set of positions which are mates for white in n moves. In ordinary chess, where each player has a finite number of possible moves at any one turn, W is simply the union of the various I^n, and W is the smallest fixed

[1] The notion of Erdös number was defined in *What is your Erdös number?* by C. Goffman, Amer. Math. Monthly **76** (1969), 791.

point of Γ. But suppose we modify the rules of chess to give the players an infinite number of legal moves, at least in certain positions. For example, imagine the board extended infinitely far to the left (i.e., $8 \times {}^*\omega$ instead of 8×8). Thus, for example, on his first move white could move his queen's rook any finite number of squares to the left. In this variant of chess, white can find himself in a winning position without having an upper bound on the number of moves he will need to win. In terms of Γ this amounts to saying that $\Gamma(\bigcup_n I^n)$ properly contains $\bigcup_n I^n$. Let $I^\omega = \Gamma(\bigcup_n I^n)$ and, more generally, $I^\alpha = \Gamma(\bigcup_{\beta<\alpha} I^\beta)$ for any ordinal α. (This agrees with the earlier definition on successor ordinals since the stages are increasing.) Then I^α might be called the set of positions where white has a win of rank $\leqq \alpha$, and $I_\Gamma = \bigcup_\alpha I^\alpha$ is the set W. It is an open problem to determine the closure ordinal of this inductive definition.[2]

This example is not as frivolous as it might appear. Chapter 4 of the book is devoted to representing inductively defined sets as the sets of winning positions in certain infinite regular two-person games. This representation theorem has important consequences for the general theory of inductive definitions. It has also led Vaught[3] to some important developments in model theory.

C. THE CANTOR-BENDIXSON DERIVATIVE. An inductive definition builds a set up from below. The dual notion is that of *coinductive definition.* A coinductive definition Γ shrinks down to a set J_Γ from above. One can move from one notion to the other by taking complements, but usually one notion is more natural than the other.

Let X be a compact, Hausdorff space. For any closed set Y let dY be the set of limit points of Y. (For other Y's let $dY = Y$.) Since d is monotone we can use it as a coinductive definition. Let $J^0 = X$ and for larger ordinals α let $J^\alpha = d(\bigcap_{\beta<\alpha} J^\beta)$. Each J^α is a closed set and $J_\Gamma = \bigcap_\beta J^\beta$ is the largest perfect closed subset of X. If X is separable, then the closure ordinal of Γ, the least

[2] The reviewer learned of these variants of chess from H. J. Keisler, who discovered the wins for white of ranks ω and $\omega+1$ illustrated below. (It is black's move; the black pieces are barred.) A. Ehrenfeucht has discovered positions of rank $\omega \cdot n$. It seems possible that there are positions of rank ω^2, but no one has written one down. The general theory of inductive definitions shows that all positions $p \in W$, have rank a recursive ordinal, but this seems far too generous. An interesting conjecture of Keisler is that if $p \in W$, then white has an effective strategy starting from p.

*	*	$\bar{K}$	*
*	*	*	*
*	P	P	Q
*	$\bar{B}$	P	P
*	*	$\bar{R}$	*
*	*	*	K
*	*	*	*
*	*	*	*

A rank ω position

*	*	$\bar{K}$	*
*	*	*	*
*	P	P	Q
*	$\bar{B}$	P	P
*	*	$\bar{R}$	$\bar{R}$
*	*	*	P
*	*	*	K
*	*	*	*

A rank $\omega+1$ position

[3] R. Vaught, *Descriptive set theory in $L_{\omega_1\omega}$*, Cambridge Summer School in Mathematical Logic, Lecture Notes in Math, Vol. 337, Springer-Verlag, New York, 1973, pp. 574–598.

α such that $J_\Gamma=\bigcap_{\beta<\alpha}J^\beta$, is a countable ordinal and, furthermore, each stage J^β loses at most a countable number of points. This analysis is what goes into the proof of the Cantor-Bendixson theorem: every separable, compact Hausdorff space is the union of a perfect closed set J_Γ and a countable set (the union of the countable collection of countable sets thrown out along the way).

D. Π_1^1 SETS OF NATURAL NUMBERS. Π_1^1 sets are sets which can be defined by a second-order formula which, in prenex form, has no second-order existential quantifiers. Kleene gave an analysis of the Π_1^1 sets over the structure $\mathbf{N}=\langle N, 0, 1, +, \cdot\rangle$ using recursion theory. Spector used Kleene's analysis to show that every Π_1^1 set over $\mathbf{N}$ is inductively definable (the converse always holds) and that the least nonrecursive ordinal is the supremum of the closure ordinals of first order inductive definitions over $\mathbf{N}$. In this way, Kleene's theory of hyperarithmetic and Π_1^1 sets can be viewed as the theory of inductive definitions over the structure $\mathbf{N}$.

This theory has had many applications in logic. It is awkward to apply directly to other fields, say algebra or topology, however, since one is not usually given a structure in terms of $\mathbf{N}$. This book makes a fresh start by developing the theory over an arbitrary structure, from the very beginning.

A survey of the book. The book is a carefully written research monograph. It is recommended reading for anyone with an interest in either model theory or definability theory (sometimes called generalized recursion theory). There is little doubt in the reviewer's mind that the book is an important chapter in the unfolding story of fragments of second-order logic.

Written in a serious, no nonsense (abstract or otherwise) style, the book follows a straight line from almost first principles to one of the frontiers of the subject. The book contains nine chapters. The first four treat that part of the theory which goes through over an arbitrary structure. The next four chapters deal with those parts of the theory which need some special assumptions about the structure until, in section 8E, the book is back where the theory began, on $\mathbf{N}$. (This section contains the nicest treatment I know of the Suslin-Kleene theorem, presented here in the form: The Δ_1^1 sets form the smallest effective σ-ring of subsets of $\mathbf{N}$.) Chapter 9 relates inductive definitions to the theory of admissible sets.

Chapter 1 contains basic definitions and discusses the closure properties of the class of inductively definable relations. For example, it is closed under $\wedge$, $\vee$, $\exists$, $\forall$ and certain forms of transfinite induction, as long as you use previously defined inductive relations in a positive way. The coinductive relations are the complements of the inductive relations. A relation which is both inductive and coinductive is called *hyperelementary*.

Chapter 2 discusses the stages of an inductive definition and proves the important Stage Comparison Theorem, which shows how to relate the stages of two different inductive definitions. It also assigns to each structure $\mathfrak{A}$ an important invariant of $\mathfrak{A}$, the ordinal $\kappa(\mathfrak{A})$ which is the supremum of the closure ordinals of the inductive definitions on $\mathfrak{A}$.

Chapter 3 is one of the main chapters of the book. It presents the basic structure theory for the inductive and hyperelementary relations. Chapter 4 proves the game-theoretic representation theorem mentioned in Example B.

In Chapter 5 the author starts chasing down results which do not hold in the complete generality of the first four chapters.

(1) *There is a hyperelementary relation which is not first order definable.*

(2) *There is an inductive relation* R *of* $n+1$ *arguments whose various sections* $R_y=\{(x_1 \cdots x_n) \mid R(y, x_1 \cdots x_n)\}$ *range over all* n*-ary inductive relations as* y *ranges over* $\mathfrak{A}$.

(3) *There is an* n*-ary inductive relation on* $\mathfrak{A}$ *which is not hyperelementary.* These are three of the results the author is after. The first tells us that we are really in the domain of second-order logic. The second has a number of important consequences, including (3). To obtain these kinds of results one needs a certain amount of coding ability not available in arbitrary structures. The notion used here is that of *acceptable* structure, a structure $\mathfrak{A}$ with a definable pairing function and a definable copy of the natural numbers. Chapters 6 and 7 go into second-order inductive definitions and second-order characterizations of the class of hyperelementary relations, still in the context of acceptable structures.

Chapter 8 restricts attention to countable, acceptable structures. The main results of the chapter are, for such structures, the following:

(4) *Every* Π^1_1 *relation is inductive, and hence every* Δ^1_1 *relation is hyperelementary.*

(5) *If* $\mathfrak{S}$ *is a* $\sum^1_1$ *definable set of relations which contains a nonhyperelementary relation, then* $\mathfrak{S}$ *contains* $2^{\aleph_0}$ *relations.*

The proof of (4) uses the game-theoretic characterization of Chapter 4; (5) uses (4). There are also some examples of other results known for **N** that do not lift to arbitrary countable acceptable structures.

In Chapter 9 the author introduces the notion of a *Spector class* of relations. The results of earlier chapters, like (2) above, show that the inductive relations on an acceptable structure form a Spector class. He then ties up the notion of Spector class with the theory of admissible sets. This chapter is an introduction to a new set of topics currently the subject of active research[4].

So much for the abstract book. The physical book deserves mention too, especially in these days, for it is beautifully produced, easy to use, and a pleasure to read. The frequency of misprints and mistakes is incredibly low. The only one worth pointing out (that the reviewer has found) is that the theorem attributed to Mansfield on p. 135 (result (5) above for the case of **N**) is due to J. Harrison; Mansfield improved the result.

Unacceptable structures. To this reviewer, the only serious objection to anything in the book is the use of acceptability. The restriction to acceptable

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[4] An important application of Chapter 9 appears in the following paper, which might well be considered as a chapter of the book: Y. N. Moschovakis, *On nonmonotone inductive definability*, Fund Math. **82** (1974) 39–83.