

Despite these shortcomings, this book provides a good introduction to the subject matter. The beginner can learn much from it and the expert can use it as a reference book. It will have its impact on the future of the field.

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Elliptic modular functions, by B. Schoeneberg, Springer-Verlag, New York, Heidelberg and Berlin, 1974, 229+viii pp.

Before the appearance of Gunning's *Lectures on modular forms* in 1962—if one leaves aside Hardy's 1940 book, *Ramanujan*, which does not attempt to deal with the theory of modular functions systematically, but instead treats the subject with the characteristically unusual (though always interesting) perspective of the great Indian mathematician in mind—the only book available in the English language in this important area of mathematics was Lester Ford's classic, *Automorphic functions*. First published in 1929 as an elaboration of a 1915 Edinburgh Mathematical Tract, Ford's book served the mathematical public well for many years. It is hardly a criticism to point out the obvious—that by the early 1960's it was long out of date. While Ford deals quite effectively with uniformization theory and with the geometry of discontinuous groups—in particular he gives a lucid account of the construction of fundamental regions for discontinuous groups by what has come to be known as “Ford's method” of isometric circles—a number of fundamental developments in the decades following the publication of Ford's book created the need for a new exposition of the theory of modular and automorphic functions in one complex variable.

Though small in size and limited in intention, Gunning's book went far toward beginning to fill this need. Treating the modular group and certain congruence subgroups from the viewpoint of the theory of compact Riemann surfaces, Gunning made available to his readers an entire complex of ideas too “modern” to appear in Ford's work. Notable examples are the application of the Riemann-Roch theorem to calculate the dimension of the space of cusp forms, the introduction of the Petersson inner product and

consequent proof that every cusp form is a linear combination of Poincaré series, and a discussion of the Hecke operators for the case of the full modular group. Gunning also applies results in the theory of modular functions to theta series and thus to the problem of determining the number of representations of an integer by a positive definite quadratic form.

Two years after the appearance of Gunning's book the American Mathematical Society published Lehner's more ambitious work, *Discontinuous groups and automorphic functions*. Broad in scope and detailed in approach, Lehner's work has had a considerable impact upon the mathematical community, as is evident from the large, and still growing, number of citations it has received in research articles since it first appeared. As Lehner is a student of Rademacher, whose school has studied modular and automorphic functions primarily from the point of view of their applications to number theory, it is no surprise that his book deals at some length with the calculation of the Fourier coefficients of automorphic forms by means of the circle method, which has its origins in such applications. (The circle method is really several different, but related, methods. The original form was devised by Hardy and Ramanujan and employed in their famous 1917 study of the partition function of number theory, which occurs as the Fourier coefficient of the modular form $1/\eta(z)$.¹ The method has since come to be known commonly as the Hardy-Littlewood method.) On the other hand, Lehner does not disregard the fundamental advances of the Hecke-Petersson school. Like Gunning he adopts the viewpoint provided by the theory of Riemann surfaces, and in particular he makes effective use of the inner product on the space of cusp forms (analogous to the inner product of abelian differentials on a compact Riemann surface), but he does so in the more general context of H -groups, finitely generated Fuchsian groups of the first kind with translations, of which the modular group and its congruence subgroups are examples. These important special cases receive a good deal of attention in Lehner's book, which devotes an entire chapter to the modular group, modular functions, and a brief sketch of several number-theoretic applications.

The period since the appearance of the books of Gunning and Lehner has been marked by a growing interest in modular and automorphic functions (even in the classical case of one complex variable), not only on the part of followers of Hecke, Rademacher, Siegel, and Selberg, but from a number of new directions as well. Not surprisingly, there has been a concomitant increase in the number of books available treating these and closely related subjects. Among these are Lehner's *A short course in automorphic functions* (1966), Andrew Ogg's *Modular forms and Dirichlet series* (1969), the book of the present reviewer, *Modular functions in analytic number theory* (1970), and Shimura's 1971 work, *Introduction to the arithmetic theory of automorphic functions*. In addition, J. P. Serre's *A course in arithmetic*

¹ See G. H. Hardy and S. Ramanujan, *Une formule asymptotique pour le nombre des partitions de n* , *Comptes Rendus* **2** (1917).

(1973) contains a substantial chapter on the theory of modular forms in one complex variable.

To the growing list may now be added Bruno Schoeneberg's *Elliptic modular functions*, published by Springer-Verlag in 1974. As the author states in the preface, "the content of the first few chapters belongs almost entirely to the repertory of every scholar in the field of elliptic modular functions Chapter VII consists essentially of an article by Hecke. Chapters VIII and IX are based on the work of the author." Despite the large number of recent books, Schoeneberg manages to present a good deal of material not to be found in any of them, testimony both to the richness and diversity of the field itself and to the importance of Schoeneberg's contributions to it over the years. It is indeed in the final Chapters 7–9 that the book makes its strongest contributions to the existing body of literature in English on modular functions of a single complex variable. Nevertheless, even the specialist may find a number of pages of interesting reading in the earlier six chapters as well. These are titled, respectively, "The modular group," "The modular functions of level one," "Eisenstein series," "Subgroups of the modular group," "Function theory for the subgroups of finite index in the modular group," and "Fields of modular functions". The final three chapters are "Eisenstein series of higher level," "The integrals of \mathcal{P} -division values," and "Theta series."

The modular group Γ is the collection of all linear fractional transformations of the form $z \rightarrow (az + b)/(cz + d)$, with a, b, c, d rational integers and $ad - bc = 1$. In very broad terms, the subject at hand is the study of the group Γ together with certain of its subgroups of finite index (the congruence subgroups) and the meromorphic invariants of such groups (the modular functions and forms). The first chapter begins with a discussion of general (complex) linear fractional transformations, quickly focuses upon the real linear fractional transformations, and then specializes to the modular group itself. Schoeneberg is careful to maintain the sometimes forgotten distinction between homogeneous and inhomogeneous linear fractional transformations, a minor point, but one whose neglect can lead to irritating difficulties in calculation.²

An important feature of Chapter 1 is the introduction of the notion of a fundamental region and the construction of a fundamental region for Γ . The uninitiated reader should be alerted that, while fundamental regions are indeed "fundamental," much of the literature on discontinuous groups does not require that they be regions. Usually the requirement of connectedness is omitted and frequently a fundamental region need not even be open. It is also the case that none of the various definitions to be found in the literature is restrictive enough to attach a unique fundamental region to a given discontinuous group. Indeed there are several ways in which uniqueness can fail with each of the definitions, and Schoeneberg's definition is no exception. However, since Schoeneberg—in contrast to Lehner, for example—deals only with subgroups of finite index in Γ , rather than with general

² In this review only the more commonly used inhomogeneous definitions are given.

discontinuous groups, he need not dwell on problems of definition. Happily, the author takes full advantage of this state of affairs, stating the definition with a minimum of fuss and immediately thereafter constructing the “standard” fundamental region for Γ , the open subset of \mathcal{H} (the upper half-plane) defined by the inequalities $|\operatorname{Re} z| < \frac{1}{2}$ and $|z| > 1$, together with half of its boundary points.

The same sensible approach is to be found in the treatment in Chapter 4 of fundamental regions for subgroups of finite index in Γ . In their construction Schoeneberg proves and applies the standard result that if Γ_1 is such a subgroup with coset decomposition $\Gamma = \bigcup_{v=1}^{\mu} \Gamma_1 S_v$ and \mathcal{F} is a fundamental region for Γ , then $\mathcal{F}_1 = \bigcup_{v=1}^{\mu} S_v(\mathcal{F})$ is a fundamental region for Γ_1 . Taking \mathcal{F} to be a “nice” set, one gets a “nice” fundamental region for Γ . In this result Γ can be replaced by any discontinuous group acting on \mathcal{H} , but the author has no need for this generality. Indeed, in Chapter 4 he is concerned primarily with $\Gamma(N)$, the *principal congruence subgroup* (of Γ) of level N , defined for N a positive integer to be the subgroup of those transformations $z \rightarrow (az+b)/(cz+d)$ in Γ satisfying the condition $a \equiv d \equiv \pm 1$, $b \equiv c \equiv 0 \pmod{N}$. (With this notation, $\Gamma(1) = \Gamma$.) A *congruence subgroup of level N* is any subgroup of Γ which contains $\Gamma(N)$. (Note that the level is not unique, but it is of interest to determine the least level of a given subgroup Γ_1 , what the author calls the *conductor* of Γ_1 .) Schoeneberg applies the above result to construct explicitly fundamental regions for $\Gamma(2)$ and indeed for all the congruence subgroups of level 2. He does the same for the congruence subgroup $\Gamma^0(q)$, of level q , with q a prime number, where $\Gamma^0(q)$ is defined by the restriction $b \equiv 0 \pmod{q}$ in the linear fractional transformation $z \rightarrow (az+b)/(cz+d)$. (There are some pleasing pictures here illustrating the fundamental regions constructed.)

The next order of business in Chapter 4 is a consideration of the parabolic cusps of \mathcal{F}_1 , those real points (necessarily rational) on the boundary of \mathcal{F}_1 . Schoeneberg gives a formula for the number of inequivalent cusps of $\Gamma(N)$, and he obtains the familiar result that $\Gamma^0(q)$, with q a prime, has exactly two cusps. Defining \mathcal{H}^* to be \mathcal{H} with the rational points (including $i\infty$) of the real axis adjoined, the author next gives a rather brief, but careful discussion of the compact Riemann surface \mathcal{H}^*/Γ_1 , whose points are orbits of \mathcal{H}^* under the group Γ_1 , a subgroup of finite index in Γ . In order to discuss the genus of \mathcal{H}^*/Γ_1 Schoeneberg quotes, without proof, Euler’s theorem on polyhedra as well as the connection between the Euler characteristic $v - e + f$ and the genus of the surface. He closes the chapter with a number of interesting and useful calculations relating to the genus of \mathcal{H}^*/Γ_1 —usually called simply the *genus of Γ_1* —for certain congruence subgroups Γ_1 of Γ . In particular he derives a formula for the genus of $\Gamma_0(N)$, a congruence subgroup of level N defined by the condition $c \equiv 0 \pmod{N}$. These groups are important in number-theoretic applications of the theory of modular forms, among other reasons, because of their connection with theta series.

Chapter 2 begins the main business of the work, the theory of modular functions and forms. A *modular function* is a function f meromorphic in \mathcal{H} which is invariant under some subgroup Γ_1 of finite index in Γ ; that is to say,

$f \circ M = f$ for all $M \in \Gamma_1$. For Schoeneberg a *modular form* is a function F meromorphic in \mathcal{H} satisfying the functional equation $(c\tau + d)^k F(M\tau) = F(\tau)$, for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$, with k a fixed integer known as the *dimension* of the form.³ (Thus a modular function is a modular form of dimension 0.) Though the author in fact has occasion—in Chapter 9—to discuss modular forms of half-integral dimensions with multiplier systems, he avoids the complications involved in giving the definition.

The characteristic functional equation of a modular form F implies the existence of a Laurent expansion in the appropriate local uniformizing variable at each parabolic cusp, if it is assumed that F has only finitely many poles in the closure of \mathcal{F}_1 . (At $i\infty$ the local variable is of the form $t = e^{2\pi iz/N}$, with N a positive integer, and at the finite cusps it is similarly an exponential; hence these Laurent expansions are often called Fourier expansions.) There is a further condition imposed upon a modular function or modular form: that the Laurent expansion be finite to the left at each parabolic cusp. If a modular form is analytic in \mathcal{H} and the expansions contain no negative powers it is called an *entire form*; if the expansions of an entire form contain only positive powers it is called a *cusp form*. Schoeneberg does not emphasize the role of the functional equation in the question of the existence of the Laurent expansion and thus, in my view, fails to motivate the discussion sufficiently.

Chapter 2 deals only with modular functions and forms of level one, that is, with the case in which Γ_1 is the full modular group Γ . The author constructs the well-known absolute modular invariant $J(\tau)$ by use of the Riemann mapping theorem and the Schwarz reflection principle, and then proves the result that contains the theoretical significance of $J(\tau)$: *The field of modular functions is the rational function field $C(J)$, with C the field of complex numbers.* Using the fact that $J'(\tau)$ is a modular form of dimension -2 , Schoeneberg then constructs all entire modular forms of level one. Singled out for special attention, as indeed they should be, are $\Delta(\tau)$, the discriminant function of elliptic function theory (a modular form of dimension -12), and those modular forms which turn out to be constructible as Eisenstein series (Chapter 3). The chapter closes with a calculation of the complex dimension of the vector space $C^+(\Gamma, -k)$ of entire modular forms (level one) of fixed integral dimension $-k$ ($C^+(\Gamma, -k)$ is trivial for k odd or $k < 0$) and with a proof that two modular forms, of dimensions -4 and -6 , suffice to generate the full graded ring of entire modular forms of negative even integral dimensions.

Chapter 3 continues the discussion of modular forms of level one, but here the construction of entire forms is effected through the use of Eisenstein series, functions of the form $G_k(\tau) = \sum (m\tau + n)^{-k}$, with k an even integer ≥ 4 , the summation being over all pairs of integers except $(0, 0)$. For $k \geq 4$ uniform convergence of $\sum |m\tau + n|^{-k}$ on compact subsets of \mathcal{H} is assured and

³ With regard to the term for k the literature is in a chaotic state. The terms “degree” and “weight” are also employed, sometimes with the same meaning as “dimension,” sometimes with a related, but different, meaning.

from this it follows without difficulty that $G_k(\tau)$ is an entire modular form of level one and dimension $-k$. The discriminant function $\Delta(\tau)$ and the absolute modular invariant $J(\tau)$ are constructed once again, this time in terms of G_4^3 and G_6^2 . Indeed, the graded ring of entire modular forms of negative even integral dimensions is generated by G_4 and G_6 . However, the real interest in the Eisenstein series resides largely in the number-theoretic significance of their coefficients in the power-series (Fourier) expansion at the parabolic cusp $i\infty$. In fact, if we write $G_k(\tau) = a_0 + \sum_{n=1}^{\infty} a_n e^{2\pi i n \tau}$, then a_0 is $2\zeta(k)$, with ζ the Riemann zeta-function, and $a_n, n \geq 1$, is essentially $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. The polynomial relations among the Eisenstein series—many of which follow immediately from a knowledge of the dimension of $C^+(\Gamma, -k)$ —lead directly to a number of interesting number-theoretic results, of which the following one derived by Schoeneberg is typical:

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{k=1}^{n-1} \sigma_3(k) \sigma_3(n-k).$$

The explicit calculation of the coefficients of G_k is usually accomplished by application of the Lipschitz summation formula, itself a consequence of the Poisson summation formula. Though Schoeneberg follows this general procedure, even mentioning Poisson by name, he unfortunately states neither summation formula explicitly. He presents Hecke's important method for dealing with G_2 —in which case absolute convergence fails—by applying the same summation techniques. There are no nontrivial elements of $C^+(\Gamma, -2)$; nevertheless, by interpreting the defining sum appropriately Hecke proved that G_2 has the transformation properties of a modular form of dimension -2 . It fails to be a modular form, however, because it is not meromorphic in \mathcal{H} ; indeed it has an expansion of the form $G_2(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n \tau} - \pi/y$, with $y = \text{Im } \tau$. From this it follows that $G_2^*(\tau) = G_2(\tau) + \pi/y$ is analytic in \mathcal{H} , but does not fulfill the transformation requirements of a modular form, instead satisfying the following, closely related, transformation law:

$$(c\tau + d)^{-2} G_2^*(M\tau) = G_2^*(\tau) + p_M(\tau), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

with $p_M(\tau)$ a rational function in τ . The Hecke method is pursued further in Chapter 7.

Chapter 5 generalizes the ideas of Chapter 2 to subgroups of arbitrary level in Γ , bringing the viewpoint of Riemann surface theory—which of necessity makes its presence felt, at least indirectly, in any discussion of modular functions—very much into the foreground. After the introduction of differentials and divisors on a Riemann surface, the Riemann-Roch theorem is stated, without proof, and applied to calculate the dimension of $C^+(\Gamma_1, -k)$, with k an even integer ≥ 2 and Γ_1 an arbitrary subgroup of finite index in Γ . Since the Riemann-Roch theorem remains unproved here (with good reason—a proof would have taken Schoeneberg far afield), I would have liked the dimension calculated by elementary means for a few congruence subgroups of low level, for which it is known that, suitably modified,

the procedure the author employs for the full group Γ can be carried through. Further sins of omission are the author's failure to give a proof of the basic Theorem 4 of Chapter 5 (*a nonconstant modular function takes on each complex value the same finite number of times in a fundamental region \mathcal{F}_1 of Γ_1*), or to state the important corollary result that *a bounded modular function is constant*. While the author's assertion that the proof of Theorem 4 is similar to that given earlier for the special case $\Gamma_1 = \Gamma$ is true enough, there remain sufficiently many differences of detail, especially in the handling of the finite parabolic cusps, to warrant a complete proof. In fairness it should be pointed out, however, that this is one of the book's very few lapses in application of the sound pedagogical principle that repetition of ideas is beneficial to the reader's understanding, especially when the context is varied. Furthermore, Schoeneberg atones by giving an elegant proof—making clever use of $\Delta(\tau)$ —that *there are no nontrivial entire modular forms of positive dimension*, and of the related formula for the number of zeros of an entire modular form of negative dimension.

Chapter 7, a natural continuation of Chapters 3 and 5, introduces the Eisenstein series $G_{N,k,a}$ of arbitrary level $N \geq 1$ and dimension $-k \leq -3$. Defined by the series $\sum (m_1\tau + m_2)^{-k}$, with summation conditions $m_i \equiv a_i \pmod{N}$ (and thus obtainable as the N th order division values of the derivatives of the Weierstrass \wp -function), the $G_{N,k,a}$, $a = (a_1, a_2)$, are entire modular forms of dimension $-k$ for the group $\Gamma(N)$. With respect to the structure of $C^+(\Gamma(N), -k)$, their importance resides in the fact that an arbitrary entire form can be written as a linear combination of Eisenstein series and a cusp form. At this point one could wish that the Petersson inner product had been introduced somewhere in the book. If it had been, it could be proved that the Eisenstein series span the orthogonal complement, within $C^+(\Gamma(N), -k)$, of the subspace of cusp forms. Of course, it is hardly to be expected that the 1927 article of Hecke which provides the material of this chapter would mention an inner product first introduced by Petersson in 1939. In this respect, at least, Schoeneberg follows Hecke too faithfully.

From the viewpoint of the specialist, the most interesting and significant portion of Chapter 7 is that which presents Hecke's analogue of the Eisenstein series of level N for the dimensions -1 and -2 . Here Schoeneberg integrates work of Hecke appearing in several different articles, and he does so with care, tastefully supplying details omitted in the original. As in the case of the full modular group Γ , the Eisenstein series of dimension -2 fail to be modular forms with respect to $\Gamma(N)$, since they are not analytic in \mathcal{H} . However, in contrast to the situation for the full group, when $N \geq 2$ there are several linearly independent Eisenstein series of dimension -2 , each with the same nonanalytic term; as a result, modular forms of dimension -2 can be constructed as the difference of two Eisenstein series. The situation for dimension -1 is different, as the nonanalytic term does not appear and thus the Eisenstein series in this case are themselves modular forms with respect to $\Gamma(N)$.

Despite the passage of almost a half-century since the work of Hecke on

Eisenstein series of dimensions -1 and -2 , the theory of automorphic forms of dimensions k , with $-2 \leq k \leq 0$, has remained largely undeveloped. Except for the integral and half-integral dimensions between -2 and 0 , the only methods of constructing automorphic (or even modular) forms of dimensions in this range have been *ad hoc* methods which afford little insight into the nature of the Fourier coefficients or the structure of the vector spaces of entire forms. (In the case of the full modular group, for example, one can take arbitrary real powers of $\Delta(\tau)$, the discriminant function, since $\Delta(\tau)$ has no zeros in \mathcal{H} .) For the case $k = -2$, Petersson has fully developed the theory of cusp forms on a congruence subgroup of Γ by generalizing to Poincaré series Hecke's method of summing Eisenstein series of dimension -2 . In addition, entire forms of dimensions $-\frac{1}{2}$ and $-\frac{3}{2}$ can be constructed through the use of theta series, but only for certain congruence subgroups of Γ . Recent work of D. Niebur, on the other hand, gives some insight into the nature of the automorphic forms of dimension 0 (automorphic functions) connected with an arbitrary H-group.⁴

As G. Bol observed in 1948, and as can be proved easily by application of the chain rule,

$$D^{(r+1)}\{(c\tau + d)^r f(M\tau)\} = (c\tau + d)^{-r-2} f^{(r+1)}(M\tau),$$

provided r is a nonnegative integer, $M\tau = (a\tau + b)/(c\tau + d)$, with $ad - bc = 1$, and the derivatives exist. Consequently, the $(r+1)$ st derivative of a modular form of dimension r is a modular form of dimension $-r-2$ and, conversely, the $(r+1)$ -fold integral of a modular form of dimension $-r-2$ is a "modular form of dimension r with period polynomials" (or Eichler integral). This latter concept is a natural generalization of the notion of an abelian integral on the Riemann surface \mathcal{H}^*/Γ , which is the special case $r=0$. In Chapter 5 Schoeneberg takes up this line of thought, but regrettably (in view of recent research activity centered upon Eichler integrals and the related Eichler cohomology groups) he does so only for the case $r=0$. He returns to the subject in a more serious way in Chapter 8, which is devoted to the calculation of the periods connected with the integrals of Eisenstein series of level N and dimension -2 . Here Schoeneberg begins by replacing the Eisenstein series by the linearly equivalent system of N th order division values of the Weierstrass \wp -function. This approach has the advantage that the latter—with one exception—are analytic in \mathcal{H} and thus modular forms of dimension -2 . The exceptional function is in fact $G_2(\tau)$, the nonanalytic Eisenstein series of level 1 discussed in Chapter 3. There Schoeneberg obtains a new construction of $\Delta(\tau)$ (the third one in the book) by integrating the modified Eisenstein series $G_2^*(\tau) = G_2(\tau) + \pi/y$ and exponentiating the result. This leads to the well-known Dedekind function $\eta(\tau)$, in fact a modular form of dimension $-\frac{1}{2}$, but not identified as such by the author, since he has restricted himself to modular forms of integral dimensions, without multipliers. (This restriction is cause for further annoyance in Chapters 8 and 9.) However, he does display the functional equation of

⁴ Niebur has now extended his work to include all dimensions k in the range $-2 \leq k \leq 0$.

$\eta(\tau)$, from which it follows that $\Delta(\tau) = \eta^{24}(\tau)$ is a modular form of dimension -12 . Schoeneberg's multifaceted treatment of the important function $\Delta(\tau)$ is, incidentally, but one example of this ability to illuminate a complex subject from several points of view, with great potential benefits to the careful reader. This, in my view, is one of the book's principal strengths.

The bulk of Chapter 8 is devoted to the calculation of the periods arising from the integrals of the \mathfrak{B} -division values. Since the transformations $Uz = z + 1$ and $Tz = -1/z$ generate the full modular group Γ and the period connected with U can be read off from the definition, the essential work is contained in calculating the periods of the integrals under the transformation T . Schoeneberg accomplishes this by two different methods, both of which have their roots in techniques developed to derive the functional equation relating $\eta(T\tau)$ to $\eta(\tau)$. The first of these is the classical method of Riemann and Dedekind, the second the relatively recent (1954) approach of C. L. Siegel, which is a subtle and clever application of the residue theorem. Exponentiating the abelian integrals under discussion, Schoeneberg obtains natural generalizations of $\eta(\tau)$ which are modular forms of dimension 0 and level N , with multipliers. These functions raised to the power $12N^2/(6, N)$ are modular functions invariant under $\Gamma(N)$.

The final Chapter 9, based upon an important 1939 article of the author, deals with theta series attached to positive definite quadratic forms in an even number of variables. (The author avoids the case of an odd number of variables since that would lead to modular forms of half-integral dimensions.) Schoeneberg derives the transformation formula for his theta series under T , and hence for an arbitrary element of Γ , the formula for U being virtually self-evident. There is an important simplification for elements of $\Gamma_0(N)$, with the level N determined by arithmetic properties of the matrix A associated with the quadratic form. (In fact, N is the least positive integer such that $N \cdot A^{-1}$ has integral entries, with even integers on the main diagonal.) For elements of the principal congruence subgroup $\Gamma(N)$ there is a further simplification; indeed the theta series are entire modular forms with respect to $\Gamma(N)$, a fact which is the principal result of Chapter 9. The proof, as could be anticipated, involves the evaluation of Gaussian sums with prime argument in terms of the Legendre symbol. This is achieved here as a corollary to the general theta transformation formula under T , derived by Poisson summation.

The chapter closes with an illustration of the general theory through several examples of level $N=1$, theta series that are modular forms with respect to the full modular group Γ . The special cases presented involve quadratic forms in 8, 16 and 24 variables, giving rise to theta series which are modular forms of dimensions -4 , -8 , and -12 , respectively. In each case the n th Fourier coefficient of the theta series (the number of representations of n by the quadratic form in question) is represented as a divisor function—arising as the n th coefficient of an Eisenstein series of the appropriate dimension—plus the n th coefficient of a cusp form on Γ . At this (rather late) point Schoeneberg presents the classical Hecke estimate for the growth

of the coefficients of a cusp form and he applies it to show that the divisor function is a reasonably good approximation to the number of representations of n by the quadratic form. Because of the restriction to $N=1$ Schoeneberg does not discuss the classical theta function, $\vartheta(\tau) = \sum_{m=-\infty}^{\infty} \exp(\pi i m^2 \tau)$, which is of level $N=2$. $\vartheta^s(\tau)$ arises from the quadratic form $x_1^2 + \cdots + x_s^2$ and thus serves as a generating function for $r_s(n)$, the number of representations of n as a sum of s squares. The author's omission of $\vartheta(\tau)$ is unfortunate in a book of this size and scope, especially since he has developed all the machinery necessary to discuss $\vartheta^s(\tau)$, at least for $s \equiv 0 \pmod{8}$, in which case $\vartheta^s(\tau)$ is a modular form of level 2 and dimension $-s/2$, without multipliers.

Undoubtedly there will be some who view Schoeneberg's book as old-fashioned. Indeed, except for Chapter 8, the book could have been written in 1939, and even the 1967 article of the author, upon which Chapter 8 is based, conceivably could have been written in 1940. My own feeling is that we should be grateful for works of this quality whenever they appear. On the other hand, I regard as a flaw Schoeneberg's failure to introduce Hecke operators or the Petersson inner product. Were the book not otherwise excellent, these omissions, in themselves, would be no cause for concern. As things are, the first six chapters constitute a well-written, solid treatment of the classical theory of modular functions of a single variable, except for the omission of these two important topics. These chapters, together with appropriately chosen additional material, could serve as an excellent year-long introduction to the subject for graduate students with a reasonable background in analysis and algebra. It is a pity that additional material must be introduced for this purpose. The book is good enough that I cannot help feeling it could have been even better, and wishing it were.

The translation, by J. R. Smart and E. A. Schwandt, is generally smooth and free of awkward phrasings. Happily, it reads like English, with little, if any, trace of the original German detectable. I noticed several misprints, but a remarkably small number for a book of this length.

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