SUBSEQUENCES OF SEQUENCES OF RANDOM VARIABLES

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Chatterji [2] has formulated the following heuristic principle: given any limit property for independent identically distributed random variables (i.i.d.r.v.'s), there exists an analogous property such that an arbitrary sequence of r.v.'s always has some subsequence possessing this analogous property. By 'arbitrary', we mean that no assumption concerning dependence is made, though it may be necessary to assume moment conditions on the r.v.'s. The purpose of this note is to announce Theorem 1, which makes this principle precise in the case where the property is an "a.s. limit theorem", a concept we formalise below.

Let $P(R)$ denote the space of probability measures on R. For $\Pi \in P(R)$, let

$$\alpha(\Pi) = \infty \qquad \text{if } \int |x| \Pi(dx) = \infty,$$

$$= \int x \; \Pi(dx) \qquad \text{otherwise,}$$

and let $\beta(\Pi) = \int (x - \alpha(\Pi))^2 \Pi(dx)$.

Define a *statute* to be a measurable subset A of $P(R) \times R^\infty$ such that, whenever $\Pi \in P(R)$ and $\{X_n\}$ is i.i.d. with distribution Π, then $(\Pi, X_1(\omega), X_2(\omega), \dots) \in A$ a.s. For example, the statutes corresponding to the strong law of large numbers and the law of the iterated logarithm are

$$A_1 = \left\{ (\Pi, x): \alpha(\Pi) = \infty \text{ or } \lim_{n \to \infty} \sum_{i=1}^{n} x_i \bigg/ n = \alpha(\Pi) \right\}.$$

$$A_2 = \left\{ (\Pi, x): \beta(\Pi) = \infty \text{ or } \limsup_{n \to \infty} \frac{\Sigma_{j=1}^{n} \; x_i - n\alpha(\Pi)}{(2n \; \log \; \log(n))^{1/2}} = \beta^{1/2}(\Pi) \right\}.$$

Here $x = (x_i)$ denotes a generic point in R^∞.

It seems clear that any "a.s. limit theorem for i.i.d.r.v.'s" may be represented by a statute. We need the following technical condition, which is satisfied by the statutes corresponding to most nontrivial such theorems.

(1) If $(\Pi, x) \in A$ and x' satisfies $\Sigma |x_i - x'_i| < \infty$, then $(\Pi, x') \in A$.

THEOREM 1. *Let $\{X_i\}$ be a tight sequence of r.v.'s defined on a probability space (Ω, F, P). Then there exists a measurable function $\mu: (\Omega, F, P) \rightarrow P(R)$ with the following property.*

Let A be any statute satisfying (1). *Then there exists a subsequence $\{X_{m_i}\}$ all of whose subsequences $\{X_{n_i}\}$ satisfy $(\mu(\omega), X_{n_1}(\omega), X_{n_2}(\omega), \ldots) \in A$ a.s.*

To apply this theorem we need a technical fact: if $\sup_i E|X_i|^p < \infty$, then $\mu(\omega)$ has finite pth moment a.s. As an example, let us apply Theorem 1 to the statute A_1 corresponding to the strong law of large numbers. We obtain the following result. *If $\{X_i\}$ is a sequence of r.v.'s such that $\sup_i E|X_i| < \infty$, then there exists $\{X_{m_i}\}$ such that*

$$\lim_{n \to \infty} n^{-1} \sum_{i=1}^{n} X_{k_i}(\omega) = \alpha(\mu(\omega)) \quad \text{a.s.}$$

for each subsequence $\{X_{k_i}\}$ of $\{X_{m_i}\}$. This gives a result of Komlós [5]. Chatterji ([2] and references therein) has proved his principle for various theorems such as the law of the iterated logarithm. All these results (except for the central limit theorem) are special cases of Theorem 1. Chatterji's technique is to reduce the case of a general sequence $\{X_i\}$ to the case of a martingale difference sequence, and then to use methods applicable to the particular theorem under consideration. Obviously this technique cannot prove a general result like Theorem 1, which applies to statutes representing theorems for i.i.d.r.v.'s that are as yet undiscovered!

The idea of our proof is to extract a subsequence which is "close to" an exchangeable sequence $\{Z_i\}$. It is essentially known that for such $\{Z_i\}$ there exists $\mu: (\Omega, F, P) \rightarrow P(R)$ such that for each statute A, $(\mu(\omega), Z_1(\omega), Z_2(\omega), \ldots) \in A$ a.s.

It has been suggested that the following result, which would immediately imply Theorem 1, might be true.

CONJECTURE. *Let $\{X_i\}$ be a tight sequence of r.v.'s. Then, by enlarging the probability space if necessary, there exists an exchangeable sequence $\{Z_i\}$ such that for some subsequence $\{X_{n_i}\}$*

$$P(|X_{n_i} - Z_i| > 2^{-i}) < 2^{-i} \quad \text{for each } i.$$

A counterexample to this conjecture will be given. A weaker result is known (Dacunha–Castelle [3]): $\{X_{n_i}\}$ can be chosen such that $(X_{n_{i+1}}, \ldots, X_{n_{i+k}}) \xrightarrow{\mathcal{D}} (Z_1, \ldots, Z_k)$ as $i \to \infty$, for each k. However, this result is not strong enough to imply Theorem 1. The proof of Theorem 1 involves choosing $\{X_{n_i}\}$ "close to" $\{Z_i\}$ in a certain, rather complicated, intermediate sense.

It is natural to look for a general result, analogous to Theorem 1, which deals with theorems about convergence in distribution. But the author can find no natural way to formulate such a result. However, the techniques used in the

proof of Theorem 1 enable the subsequence principle for Donsker's Theorem (see Billingsley [1]) to be proved, and this is the deepest "convergence in distribution" theorem that the author knows.

These techniques also yield a proof of the following result stated by Révész [6, Theorem 5.1.1], whose proof is fallacious, as observed in [4].

THEOREM 2. *Let* $\{X_i\}$ *be a sequence of r.v.'s such that* $\sup_i E\,X_i^2 < \infty$. *Then there exists a subsequence* $\{X_{n_i}\}$ *and a r.v.* α *such that*

$$\sum_{i=1}^{\infty} a_{\sigma(i)}(X_{n_{\sigma(i)}} - \alpha)$$

converges a.s. for each sequence $\{a_i\}$ *of constants such that* $\Sigma\,a_i^2 < \infty$ *and for each permutation* σ *of the positive integers.*

Proofs of these results will appear elsewhere.

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