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An algebraic introduction to mathematical logic, by Donald W. Barnes and John M. Mack, Graduate Texts in Mathematics, no. 22, Springer-Verlag, New York, Heidelberg, Berlin, 1975, viii + 121 pp., $10.80.

An outline of mathematical logic, by Andrzej Grzegorczyk, Synthese Library, vol. 70, D. Reidel Publishing Co., Dordrecht, Holland and Boston, 1974, x + 596 pp., $45.00 (cloth), $24.00 (paper).

Completeness, compactness, and undecidability, by Alfred B. Manaster, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1975, vi + 154 pp.

Mathematical logic, by Daniel Ponasse, Gordon and Breach Science Publishers, New York, 1973, x + 126 pp., $12.50.

Choosing a new textbook is like buying a house. There is no hope of getting just what you want unless you create it yourself. It is usually impossible to find a book that covers exactly the material you want in the way you want it covered. What the instructor has to settle for is, at best, a book with a solid foundation on which he can build his kind of course, a book covering the basic material in what he considers a reasonable way but with enough flexibility to be expanded and remodelled here and there without disaster. The purpose of this review is to help someone choosing a new text for a first course in mathematical logic decide whether any of the above books provide a foundation for his kind of course.

What must be covered in a first course in mathematical logic? Let us presuppose a one semester course at the advanced undergraduate or beginning graduate level, a course aimed at future users of mathematics, not just future logicians. After such a course the student should be aware of the basic notions and results, both those with applications to other branches of mathematics, but also those which have something important to say about the nature of mathematics. First and foremost, the student should leave the course with a working knowledge of what mathematical concepts and notions are expressible in first-order logic, either directly or indirectly (within, say, axiomatic set theory). Without this, the rest of the course is pointless. He should learn what the Gödel Completeness Theorem has to say about the mathematicians informal notions of "proof" and "provable". The Completeness and Löwenheim-Skolem Theorems are essential for grasping the strengths and weaknesses of first-order logic, and for applications. Finally, the student should understand the Gödel Incompleteness Theorems and what they say about the nature of mathematics. This is the hard core of any reasonable first course in mathematical logic. It provides a minimal knowledge for anyone working in modern pure mathematics. Let us see how the four books listed above cover this basic hard core.

1. The Barnes and Mack book. This is a short, straightforward book which assumes a fair amount of algebraic sophistication from the student. It would not be appropriate for an undergraduate course but could be used with wellprepared graduate students. It is the strongest of the books in giving the students a feeling for what can be done in first-order logic, with lots of examples of first-order theories and possible applications. It takes a "Hilbert style proof" approach to the Completeness Theorem. It treats the Compactness and Löwenheim-Skolem Theorems, but the notion of elementary extension is not made explicit. The first Gödel Incompleteness Theorem, and the undecidability of first-order logic, are proved by representing the action of Turing machines within a finite fragment of arithmetic. It also contains short discussions of elimination of quantifiers, ultraproducts, nonstandard analysis and Hilbert's Tenth Problem, discussions which could be either expanded or eliminated.

By and large this seems a reasonable book for a graduate course. Its chief drawback is its very heavy-handed algebraic approach to trivial syntactic considerations. For example, the notions of formula and sentence are never explicitly defined! The reason is that what passes here for a formula is an element of a set $P(V,\mathfrak{R})$ defined as a certain free algebra $\tilde{P}(V,\mathfrak{R})$ factored out by a moderately complicated congruence relation. V is a set of variables (there is no distinction between constant symbols and variables here) and $\mathfrak{R}$ is a set of relation symbols. The authors obviously feel that this approach has much to recommend it. To the reviewer, it is the principal disadvantage of the book, the one which caused him to largely ignore the book when it was the text for a graduate course. After all, one of the points of logic is that there is an advantage to considering syntactic expressions as mathematical objects. The sooner the student learns this, the better. On the other hand, there is no doubt that a lot of mathematicians would be happier manipulating elements of $\tilde{P}(V,\mathfrak{R})/\approx$ than manipulating "formulas".

2. The Grzegorczyk book. This book covers pretty much the same material as the one discussed above, but it takes 596 pages as compared with 121. The flavor of the book is well captured by its subtitle: *Fundamental results and notions explained with all details*. The pace is very leisurely, with lots of discussion. It would be excellent summer-time reading for someone going off to graduate school in mathematics, but the pace would make it hard to use in a course. There are not as many examples of first-order theories and applications as in the first book.

3. The Manaster book. This is a short, straightforward book aimed at a much less advanced audience than the first book. Because it attempts to present the material to students with no experience in abstract algebra, it has to take most of its examples from the structure of the natural numbers and from English, ignoring the pitfalls which lurk in natural languages. The approach to the Completeness Theorem is via a Gentzen style calculus. It takes Buchi's approach to the Incompleteness Theorem, assigning to each Turing machine T and each natural number $\mathbf{n}$ a formula $\mathcal{Q}_{T,\mathbf{n}}$ which has a model if and only if T does not halt on input $\mathbf{n}$. This proof is then modified to get the undecidability of arithmetic.

In the reviewer's opinion, based on painful experience (though not with this particular book) it is simply a mistake to try to teach this material to students without some background in abstract mathematics. They can not appreciate what is going on. For students *with* such a background, this could be a useful book, but it should be supplemented with many, many more examples and applications. Of course this is exactly the kind of thing a teacher can and should do.

4. The Ponasse book. This book concentrates almost exclusively on the Completeness Theorem and its relation with Boolean algebra and general topology. The Compactness and Löwenheim-Skolem Theorems are there, but never discussed or used. There are almost no examples or applications, and the Incompleteness Theorems are not mentioned. What the book *does* cover could be useful to a student of logic, but it would be an unfortunate way to introduce mathematics students to the basic concerns of mathematical logic.

JON BARWISE

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Introduction to spectral theory: *Selfadjoint ordinary differential operators*, by B. M. Levitan and I. S. Sargsjan, Translations of Mathematical Monographs, vol. 39, Amer. Math. Soc., Providence, Rhode Island, 1975, xi + 525 pp. $52.80.

This book gives a systematic account of a number of basic topics in the modern spectral theory of selfadjoint ordinary differential operators, particularly second order and a system of two first order operators. It also contains, in substantially less detail, the spectral theory concerning nth order operators and is simply meant to serve as an introduction to their area of study.

A differential operator is said to be regular if the domain of its variables is finite and its coefficients are continuous. If the domain is infinite and/or all or some of the coefficients are not summable, then the differential operator is called singular. The basic spectral theory of regular second order differential operators consists of the Sturm-Liouville theory and much space is devoted in this book to regular problems. Nonetheless, the principal content of the book is the spectral theory of singular operators. This theory was founded by H. Weyl whose work, together with the classical moment problem, played an important role in the development of a general spectral theory of symmetric and selfadjoint operators, through the efforts of F. Riesz, J. Von Neumann and others. H. Weyl's remarkable result on the limit circle and limit point gives a complete description for symmetric second order differential operators and of all its selfadjoint extensions. The general problem of describing all selfadjoint extensions of a symmetric operator was solved later by J. Von Neumann. A large role in popularizing the spectral theory of differential operators was played by the monographs of E. C. Titchmarsh, in which a new approach to the theory of singular second order operators was given. Much space is allotted in this book to singular systems of two first order operators also.

Although it appeared at the beginning that the abstract spectral theory