to introduce the states also, and develop the theory along lines in which both the logic and the space of its states play equally fundamental roles. The phenomenon of complementarity and the problems connected with the existence of a lattice structure on a logic then appear to emerge more clearly out of the manner in which the observables and states are interconnected. The expositions of Mackey and Zierler are of this type. To dismiss one of the crucial aspects of the subject in such a perfunctory manner as Piron has done is, at the least, very misleading. I would also like to point out that Piron makes no reference to the work of Zierler on the characterization of standard logics, although Zierler's work was done more or less simultaneously with Piron's and independently of it. There are many such instances of a lack of proper care in giving references to the work of others scattered throughout this book, making this exposition somewhat distorted. The reader who wants to be informed in depth on the various aspects of the subject and the extensive literature on these questions would do well to consult the volume entitled *The logico-algebraic approach to quantum mechanics*, edited by C. A. Hooker (D. Reidel Publishing Company).

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Non-archimedean fields and asymptotic expansions, by A. H. Lightstone and A. Robinson, North-Holland Mathematical Library, vol. 13, North-Holland/American Elsevier, Amsterdam and New York, 1975, 204 + x pp., \$24.95.

The aim of this short book is to show that nonarchimedean fields and nonstandard analysis form an excellent setting for the study of asymptotic expansions. The authors have been quite successful in achieving this goal. One wishes there were more, but the terminal illness of Abraham Robinson, who wrote the first draft, prevented further collaboration on Harald Lightstone's final manuscript. Since both authors are now deceased, it will be up to others to further their ideas.

An asymptotic expansion for a function f with respect to an "asymptotic sequence" of functions ϕ_i is a formal series $\sum_{i=0}^{\infty} a_i\phi_i$ such that while the sequence of partial sums $S_n(x) = \sum_{i=0}^{n} a_i\phi_i(x)$ may diverge at a given x, there may yet exist an n_x (in practice, small) such that $S_{n_x}(x)$ is a satisfactory approximation to $f(x)$. For the most part, the book deals with real, rather than complex, valued functions. A sequence of real-valued functions $\phi_0, \phi_1, \ldots$ is called an asymptotic sequence if there is a neighborhood of $+\infty$ in the real line R on which each ϕ_i is defined and nonvanishing and for each n in the natural numbers $N = \{0, 1, 2, \ldots\}$ we have $\phi_{n+1} = o(\phi_n)$, i.e., $\lim_{x\to\infty} \phi_{n+1}(x)/\phi_n(x) = 0$. Given an asymptotic sequence $\{\phi_i\}$, a sequence of real numbers $\{a_i\}$, and a real-valued function f defined on some neighborhood $(t, +\infty)$ of $+\infty$ in R, the formal sum $\sum a_i\phi_i$ is called an asymptotic expansion for f, and we write $f \sim \sum a_i\phi_i$, if for each $n \in N$, $f - \sum_0^n a_i\phi_i = o(\phi_n)$. One may think of the nth error as being a higher order of infinitesimal than the last term adjoined to the series.

For the most part, the book deals with asymptotic expansions of the form $\sum_{i=0}^{\infty} a_i x^{-\nu_i}$ where $\{\nu_i\}$ is an unbounded, strictly increasing sequence in R. For $\nu_i = i$ we have an asymptotic power series; these behave very much like ordinary power series and have similar applications such as to solutions of differential equations.

The formal development of asymptotic expansions begins in the book's sixth chapter, while Chapter 5 contains examples derived from integration by parts and the Euler-Maclaurin expansion, discussed in detail, of integrals of the form $\int_n^m g(t)\,dt$, m, $n \in N$. Typical examples are the incomplete factorial function ei, where for $t > 0$,

$$ei(t) \equiv \int_t^\infty \frac{e^{-x}}{x}\,dx \sim e^{-t}\left[\frac{1}{t} - \frac{1}{t^2} + \cdots + (-1)^n \frac{n!}{t^{n+1}} + \cdots\right],$$

and the exponential integral

$$E_i(t) \equiv e^t \int_t^\infty \frac{e^{-x}}{x}\,dx \sim \sum_{n=0}^{\infty} (-1)^n \frac{n!}{t^{n+1}}.$$

Of course, $\sum (-1)^n n!/t^{n+1}$ diverges, but for $k + 1$ equal to the integral part of t,

$$S_k(t) = e^{-t} \sum_{n=0}^{k} (-1)^n n!/t^{n+1}$$

is a good approximation to $ei(t)$; moreover, the larger the value of t, the better the approximation. For example, the value of $ei(5)$ to five decimal places is .00115, while $S_4(5) = .001173$, $S_5(5) = .001121$, and $S_6(5) = .001183$. On the other hand, $|ei(10) - S_9(10)| < 1.7 \times 10^{-9}$.

To consider in what sense a possibly diverging sequence $\sum_{i=0}^{n} a_i x^{-\nu_i}$ converges to a function f, the authors turn to the Popken space P [**5**] in Chapter 7. The space P consists of equivalence classes of asymptotically finite functions, i.e., real-valued functions f which are each defined in some neighborhood of $+\infty$ and are dominated there in absolute value by Bx^t for some B and t in R. As usual, we write $f = O(x^t)$. Two such functions represent the same element of P, and we write $f \cong g$, if $|f - g| = O(x^{-n})$ for each $n \in N$; e.g., $e^{-x} \cong 0$.

Popken defined a nonhomogeneous norm ϕ for P by setting $\phi(f) = e^\lambda$ where $\lambda = \inf\{t \in R: f = O(x^t)\}$. Given asymptotically finite functions f and g and $c \neq 0$ in R, $\phi(f) = 0$ iff $f \cong 0$, $\phi(-f) = \phi(f)$,

$$\phi(f + g) \leqslant \max[\phi(f), \phi(g)] \leqslant \phi(f) + \phi(g),$$

$\phi(f \cdot g) \leqslant \phi(f) \cdot \phi(g)$, and $\phi(cf) = \phi(f)$. As usual, one defines a metric d on P by setting $d([f],[g]) = \phi(f - g)$. As a metric space, P is complete. A series $\sum_{n \in N} f_n$ converges in P if and only if $\lim_{n\to\infty} f_n = 0$. Here one thinks of x^{-t} as a higher order infinitesimal than x^{-s} when $0 < s < t$; if $\lim_{n\to\infty} f_n = 0$, then later terms in the series $\sum f_n$ are higher order infinitesimals than the previous terms and can add essentially nothing to their sum. Given an asymptotically finite function f and a strictly increasing unbounded sequence $\{\nu_i\}$ in R, $f \sim \sum_{i \in N} a_i x^{-\nu_i}$ if and only if $f = \lim \sum_{i=0}^{n} a_i x^{-\nu_i}$ in P.

Infinitesimals are usually considered as elements of some nonarchimedean field, that is, an ordered field F containing an element a such that for every $n \in N$, the multiplicative identity added to itself n times is smaller than a. Such an element or its negative is called an infinite element of F; 0 and the multiplicative inverses of infinite elements together form the infinitesimal elements of F.

The space L' of formal sums $\sum_{n \in N} a_n x^{-\nu_n}$, where $\nu_n < \nu_{n+1}$ for each n and $\lim_n \nu_n = +\infty$, is a nonarchimedean field with the obvious definition of multiplication and addition. If some $a_n \neq 0$, we may assume $a_0 \neq 0$. The positive elements of L' are those for which $a_0 > 0$. For example, $x - n$ is positive for each $n \in N$ and x^{-1} is infinitesimal. If one writes $\sum a_n x^{\nu_n}$ instead of $\sum a_n x^{-\nu_n}$, one obtains the space L studied by Levi-Civita [**2**] in the late nineteenth century, A. Ostrowski [**4**] in the 1930s, and D. Laugwitz in 1968 [**1**].

The authors discuss L at the end of Chapter 1 as an application of the theory of nonarchimedean valuations and the corresponding metrics developed there. A nonarchimedean valuation on a field F is a map $v\colon F \to R \cup \{+\infty\}$ such that $v(0) = +\infty$, $v(x) \in R$ for all $x \neq 0$, and for all $x, y \in F$, $v(x \cdot y) = v(x) + v(y)$ and $v(x + y) \geqslant \min[v(x), v(y)]$. The corresponding metric d on F is derived from the function $|x|_v = e^{-v(x)}$ by setting $d(x,y) = |x - y|_v$. It follows that $|0|_v = 0$, $|1|_v = 1$, and for every x and y in F, $|x|_v = |-x|_v$, $|xy|_v = |x|_v|y|_v$, $|x + y|_v \leqslant \max[|x|_v, |y|_v] \leqslant |x|_v + |y|_v$, and $|y|_v < |x|_v \Rightarrow |x + y|_v = |x|_v$. In L', for example, one sets $v(0) = 0$ and $v(\sum_{n \in N} a_n x^{-\nu_n}) = \nu_0$ if $a_0 \neq 0$. As a metric space, L' is complete.

There are a number of interesting properties of these special metric spaces. For example, if $\lim a_n = a$ and $a \neq 0$, then for all sufficiently large $m \in N$, $|a_m|_v = |a|_v$. If the space is complete, $\sum_{n \in N} a_n$ converges if and only if $\lim_{n \to \infty} a_n = 0$. These and similar results are carefully established throughout the book with, unfortunately, no greater emphasis than that given to very familiar results. For example, a proof (Theorem 1.5.1) using special characteristics of $|\cdot|_v$ is given for the fact that if $\lim S_n = S$ then $\lim |S_n|_v = |S|_v$; but of course, $||S_n|_v - |S|_v| \leqslant |S_n - S|_v$. In Chapter 4, the authors discuss a nonarchimedean field ${}^\rho R$, and what they call S-continuity (Definition 4.4.1) is the ordinary notion of continuity of a function from a metric space into itself. More than a page is devoted to showing that $f\colon {}^\rho R \to {}^\rho R$ is S-continuous at $a \in \operatorname{Dom} f$ if and only if for any sequence $a_n \to a$, $f(a_n) \to f(a)$ (Theorem 4.4.6.). Of course these eccentricities, as well as the few factual errors (e.g. Lemmas 1.5.12, 3.5.6, 4.5.5), will be at most an annoyance to the careful reader.

The Popken space P is not a field, but there is a sense in which asymptotically finite functions take their values in a nonarchimedean field. To explore this idea we turn to Abraham Robinson's nonstandard analysis [**6**].

Let $\mathfrak{R}$ be the set theoretic structure built up from R; each object in $\mathfrak{R}$ is obtained from R in a finite number of steps using the usual operations of set theory. For example, the set of all Borel measures on R is in $\mathfrak{R}$. The book refers to $\mathfrak{R}$ as a many sorted structure.

Let $\mathfrak{L}$ be a formal language for $\mathfrak{R}$; $\mathfrak{L}$ contains a name for each object in $\mathfrak{R}$, variables, connectives (i.e., $\neg$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$), quantifiers, and an assortment of brackets. The language $\mathfrak{L}$ also contains sentences built up from these "atomic" symbols. One shows that there exists a structure $^*\mathfrak{R}$ built up from a set of individuals $^*R$ with the following properties:

(1) Every name of an object in $\mathfrak{R}$ names something of the same type (i.e. built up with exactly the same operations from $^*R$) in $^*\mathfrak{R}$.

(2) (Transfer Principle) Every sentence in $\mathfrak{L}$ that is true for $\mathfrak{R}$ is true when interpreted in $^*\mathfrak{R}$; quantifiers, however, must be correctly interpreted.

(3) There is a $\gamma \in {}^*N$ such that $1 < \gamma, 2 \leqslant \gamma, \ldots$; i.e., γ is an infinite "integer".

If A is in $\mathfrak{R}$, then the object in $^*\mathfrak{R}$ with the name A is denoted by $^*A$. Either A or $^*A$ can be called standard; $^*A$ is also called the nonstandard extension of A. One can omit the star for real numbers; in this sense, $R \subset {}^*R$.

The transfer principle gives us a great deal of information about $^*\mathfrak{R}$. One cannot, however, describe in a formal way what one means by all subsets of even the natural numbers; "all subsets" is a primitive notion. In $^*\mathfrak{R}$, therefore, we can cheat. If A is an infinite set in $\mathfrak{R}$, and $P(A)$ is its set of all subsets in the usual sense, then $^*(P(A)) \subsetneqq P(^*A)$. An object a in $^*\mathfrak{R}$ is called internal if there is some object A in $\mathfrak{R}$ such that $a \in {}^*A$, otherwise a is called external. In $^*R$, the symbol $\forall$ is interpreted as meaning for all internal $\ldots$, and $\exists$ is interpreted as meaning there exists an internal $\ldots$. The set of infinite natural numbers, for example, is external.

Chapter 2 gives an introduction to the nonstandard real numbers using an ultrapower construction. Ultrapowers have the additional property that every sequence $a_1, a_2, \ldots$ in the extension $^*A$ of a standard set A is the restriction to N of an internal mapping from $^*N$ into $^*A$. In an ultrapower the ε-relation is not the usual one. However, starting with "sets" of individuals and replacing each such object A with the set of all individuals a such that $a \in A$ in the sense of the ultrapower, one can form a set theoretic structure in the usual sense. (See [**6**, pp. 28–29].)

Chapter 2 is a fairly good introduction to the nonstandard real numbers. However, the paragraph on p. 42 concerning what the authors call internal and external languages is very misleading. Statements that should be called standard such as "$5 \in {}^*N$" are called internal, and some statements that should be called internal such as "$\gamma \in {}^*N$", where γ is infinite, are called external.

Let $^*\mathfrak{R}$ be a nonstandard model for the real numbers in the sense described above; $^*\mathfrak{R}$ is a nonarchimedean field. If $a \in {}^*R$ is not infinite, then a differs from a unique standard $b \in R$ by an infinitesimal. In this case, we write $a \simeq b$ and $b = {}^0a$. If f is a real-valued function on R, $^*f$ denotes the extension of f to $^*R$.

Fix a positive infinitesimal $\rho \in {}^*R$. Let M_0^ρ or just M_0 be the set $\{t \in {}^*R: |t| < \rho^{-n}$ for some $n \in N\}$, and let M_1^ρ or just M_1 be the set $\{t \in {}^*R: |t| < \rho^n$ for every $n \in N\}$. One calls a and b in M_0 equivalent and writes $a \approx b$ if $a - b \in M_1$. The set of equivalence classes ${}^\rho R = M_0 - M_1$, discussed in Chapter 3, is a nonarchimedean field with nonarchimedean valuation v_ρ given by $v_\rho(0) = +\infty$ and

$$v_\rho(\alpha) = {}^0(*\log_\rho|x|)$$

for $\alpha \neq 0$ and $x \in \alpha$. Both L and L' are isomorphic to a subfield of ${}^\rho R$; for L use the map $\sum_N a_n x^{\nu_n} \to \sum_N a_n[\rho]^{\nu_n}$. If f is real-valued and sufficiently smooth on R, then whenever $x \approx y$ in M_0, $*f(x) \approx *f(y)$ in M_0. In this case, there is a natural extension of f to ${}^\rho R \supset R$ and thus to $L \supset R$. These extensions are explored by Robinson in [9], which is also an excellent introduction to nonarchimedean fields in general and L and ${}^\rho R$ in particular.

To return to the Popken space P, let $*\hat{R}$ denote the infinite positive elements of $*R$. A function f is asymptotically finite if and only if $*f(\gamma) \in M_0^{1/\gamma}$ for each $\gamma \in *\hat{R}$. If f and g are asymptotically finite and equivalent (i.e., $|f - g| = O(x^{-n})$ for each $n \in N$), then $|*f(\gamma) - *g(\gamma)| \in M_1^{1/\gamma}$ for each $\gamma \in *\hat{R}$. Thus, up to equivalence, each $f \in P$ takes "its" value $*f(\gamma)$ in ${}^{1/\gamma}R$ for each $\gamma \in *\hat{R}$. Moreover, the Popken norm ϕ can be obtained by setting

$$\phi(f) = \sup_{\gamma \in *\hat{R}} |*f(\gamma)|_{v_{1/\gamma}}.$$

In analysis, standard definitions can often be given a quite different and more intuitive form using nonstandard notions. For example, a standard function f with domain $D \subset R$ is continuous at $a \in D$ if and only if for each $x \in *D$ with $x \simeq a$, $*f(x) \simeq f(a)$. There are a large number of such applications of $*R$ to asymptotic expansions in the last two chapters of the book. Again, one wishes there were much more. Nonstandard analysis has progressed far beyond this stage in many areas of mathematics. It will be up to others, however, to make that kind of progress here based on the foundation established by Lightstone and Robinson.

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