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# BOOK REVIEWS

*The wave equation on a curved space-time*, by F. G. Friedlander, Cambridge Monographs on Mathematical Physics, no. 2, Cambridge Univ. Press, New York and London, 1976, x + 282 pp., $39.50.

The geometrical theory of Cauchy's problem for linear hyperbolic partial differential equations with variable coefficients was originated by Hadamard and, years later, reshaped by Marcel Riesz. Now F. G. Friedlander in the book under review incorporates both Hadamard's and Riesz's ideas in a new formulation of the geometric theory in terms of distributions.

To discuss this subject, we begin with Hadamard. Summarizing his work on Cauchy's problem in a lecture series in 1920 [5], [6], Hadamard stressed the underlying analogy between elliptic and hyperbolic equations, and, through several major innovations, showed how Green's formula can be adapted from its original context of potential theory to Cauchy's problem for the wave and other hyperbolic equations. Both the analogy Hadamard referred to, and the obstacles to applying Green's formula to hyperbolic equations in the way he intended, are apparent when we compare, for instance, the three-dimensional Laplacian

$$\Delta_3 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$$

and the three dimensional d'Alembertian

$$\Box_3 = \partial^2/\partial x_2^2 - \partial^2/\partial x_2^2 - \partial^2/\partial x_3^2.$$

Let $\xi = (\xi_1, \xi_2, \xi_3)$ denote any fixed point, and $x = (x_1, x_2, x_3)$ a variable point, in real three-dimensional Euclidean space $E^3$. If $d(\xi, x)$ is the Euclidean distance $\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2}$, and $r(\xi, x)$ the Lorentzian distance $\sqrt{(x_1 - \xi_1)^2 - (x_2 - \xi_2)^2 - (x_3 - \xi_3)^2}$, between $x$ and $\xi$, then Laplace's equation $\Delta_3 u = 0$ has the solution $U_3(\xi, x) = 1/d(\xi, x)$ for $d(\xi, x) > 0$ and the homogeneous wave equation $\Box_3 v = 0$ the solution $V_3(\xi, x) = 1/r(\xi, x)$ for $r(\xi, x)$ real and $> 0$. The first function $U_3(\xi, x)$ enters the theory of Poisson's equation $\Delta_3 u = f$. In fact, if $u(x)$ is a solution of the equation on, say, a finite, smoothly bounded domain $\Omega$, then a representation of $u(\xi)$, for any $\xi \in \Omega$, is obtained by using $U_3(\xi, x)$ as auxiliary function in Green's formula on the subregion of $\Omega$ on which $d(\xi, x) > \varepsilon$ and then letting $\varepsilon \downarrow 0$. Hadamard's theory made it possible to use $V_3(\xi, x)$ in an analogous procedure pertaining to the inhomogeneous wave equation $\Box_3 v = f$ say on the half-space $E^3_+ = \{x = (x_1, x_2, x_3): x_1 > 0\}$. The eventual outcome of that procedure is a representation of $v(\xi)$, for any $\xi \in E^3_+$, in terms of the values of $v$, $\partial v/\partial x_1$ and their derivatives with respect to $x_2, x_3$, on the initial surface $x_1 = 0$. The first step in Hadamard's method is to use $V_3(\xi, x)$ as auxiliary function in an integral formula analogous to Green's formula, but pertaining to the d'Alembertian rather than the Laplacian. We refer to all such integral formulas simply as Green's formula. In the case at hand, this formula is applied on the domain $D^{\xi}(\varepsilon) = \{x = (x_1, x_2, x_3)\colon 0 < x_1 < \xi_1, r(\xi, x) > \varepsilon\}$ in an obvious parallel to the first step for Poisson's equation. The second step, having $\varepsilon \downarrow 0$, is, however, far more difficult. This is because, as $\varepsilon \downarrow 0$, the nonhorizontal part of the boundary of $D^{\xi}(\varepsilon)$ approaches a nondegenerate surface, the portion of the back characteristic cone $\{x = (x_1, x_2, x_3)\colon x_1 < \xi_1, r(\xi, x) = 0\}$ situated between its vertex $\xi$ and the initial plane $x_1 = 0$. For this reason, the integrals that occur in Green's formula on $D^{\xi}(\varepsilon)$ and its boundary need not all be expected to converge as $\varepsilon \downarrow 0$; in higher-dimensional problems, none of the corresponding integrals need converge. Hadamard dealt with this apparently disastrous fault by inventing a new concept in analysis, that of the "finite part" of a divergent integral. In an odd-dimensional space like $E^3$, Hadamard noticed that an integral such as the one just indicated over the region $D^{\xi}(\varepsilon)$ has an expansion in fractional powers of $\varepsilon$, at most a finite number of the powers being negative; similarly for the integrals in Green's formula on the boundary of $D^{\xi}(\varepsilon)$. The "finite part" of one of these integrals is the part in this expansion with respect to $\varepsilon$ that remains finite as $\varepsilon \downarrow 0$, and the relation expressed by Green's formula among the original divergent integrals implies a relation among the finite parts of the integrals. By calculating the finite parts, Hadamard was able to obtain the representation of $v(\xi)$ referred to notwithstanding the divergencies as $\varepsilon \to 0$.

Any $m$-dimensional linear differential operator $L = a^{ij}(\partial/\partial x^i)(\partial/\partial x^j)$ reducible to the d'Alembertian

$$\partial^2/\partial y_1^2 - \partial^2/\partial y_2^2 - \cdots - \partial^2/\partial y_m^2$$

by a linear change of independent variables $y = \lambda(x)$ is of course subject to the theory of the d'Alembertian. (We use the summation convention, under which repeated indices are summed from 1 to $m$.) Hadamard calls such an operator "hyperbolic"; for $L$ to be hyperbolic it is necessary and sufficient that the quadratic form $a^{ij}t_i t_j$ be of normal hyperbolic type, i.e., be reducible to $s_1^2 - s_2^2 - \cdots - s_m^2$ by means of a linear substitution $t_i = c_{ij}s_j$. If $x = (x^1, \ldots, x^m)$ and $\xi = (\xi^1, \ldots, \xi^m)$ it turns out with reference to the Lorentzian distance between $y = (y_1, \ldots, y_m) = \lambda(x)$ and $\eta = (\eta^1, \ldots, \eta^m) = \lambda(\xi)$ that

$$(y_1 - \eta_1)^2 - (y_2 - \eta_2)^2 - \cdots - (y_m - \eta_m)^2 = a_{ij}(x^i - \xi^i)(x^j - \xi^j),$$

the matrix $(a_{ij})$ being the inverse to the coefficient matrix $(a^{ij})$. In the case $m = 3$, for instance, the function $[a_{ij}(x^i - \xi^i)(x^j - \xi^j)]^{-1/2}$ thus is a solution of the equation $Lv = 0$ for such $x$ that ($\xi$ being fixed) the denominator is real and positive, and this function can be used in Hadamard's process directly without going to the trouble of changing variables to reduce $L$ to the d'Alembertian. Such observations suggest the geometrical concepts Hadamard introduced to treat hyperbolic equations with variable coefficients.

An $m$-dimensional linear differential operator $M$ defined by

$$Mv \equiv \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial v}{\partial x^j}\right) + b^i \frac{\partial v}{\partial x^i} + cv,$$

with coefficients that are permitted to be functions of $x$, is called hyperbolic if the quadratic form $g^{ij}t_it_j$ is of normal hyperbolic type for each $x$. Here $g = |\det(g_{ij})|$ with $(g_{ij}) = (g^{ij})^{-1}$. Hadamard extended the foregoing methods to the operator $M$ by use of the "pseudo-Riemannian" manifold having the first fundamental form $d\sigma^2 = g_{ij}dx^i dx^j$. The geodetic distance between two points of this manifold takes the place of the Lorentzian distance. Along certain geodesics of this manifold, $d\sigma^2 = 0$. These are called "null geodesics", the geodetic distance between two points of a null geodesic being zero. The conoid consisting of all null geodesics issuing from a specified vertex $\xi$ is the analog in this context to the characteristic cone with vertex $\xi$; it is called a "characteristic conoid". In what Hadamard named Cauchy's problem, a solution $v(x)$ of $Mv = f$ is sought having prescribed values and prescribed normal derivatives on a given $(n-1)$-dimensional "initial surface" $S$. The prescribed functions are called "Cauchy data". The initial surface is required to be "space-like", which means that $d\sigma^2 < 0$ on $S$ and implies that $S$ truncates any characteristic conoid issuing from a nearby vertex $\xi$, $S$ and the conoid together enclosing a compact $m$-dimensional region $D_S^\xi$. The boundary of $D_S^\xi$ consists of a portion $S^\xi$ of $S$ and a portion $C_S^\xi$ of the conoid. The effect of Hadamard's theory is to represent $v(\xi)$, for $\xi$ sufficiently near $S$, by means of certain integrals over $S^\xi$ and its boundary involving Cauchy data and over $D_S^\xi$ and $C_S^\xi$ involving $f$. To derive this representation, a function $W_m(\xi, x)$ is constructed that satisfies inside $D_S^\xi$ the adjoint homogeneous equation

$$M^*w \equiv \frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial x^i}\sqrt{g}\, g^{ij}\right)\frac{\partial w}{\partial x^j} - \frac{\partial}{\partial x^i}(b^i w) + cw = 0$$

and is singular on $C_S^\xi$ in a manner similar to that of the auxiliary function for the d'Alembertian. This function and $v(x)$ are used in Green's formula on a region such as $D_S^\xi(\varepsilon) = \{x \in D_S^\xi\colon |W_m(\xi, x)| > \varepsilon\}$. Then the desired representation of $v(\xi)$ results in spaces of odd dimension $m$ by taking the finite parts of the integrals involved as $\varepsilon \downarrow 0$. In even-dimensional spaces, a parallel procedure can be employed, but with what is called the "logarithmic part", not the finite part, of the integrals in question.

Powerful and, in concept, elegant, Hadamard's method was intimidating in its details. For this and other reasons, Marcel Riesz in 1949 [8] presented an alternative way of accomodating Green's theorem to the markedly singular behavior of Hadamard's auxiliary functions. To describe Riesz's method in the case of the d'Alembertian

$$\Box = \Box_m = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_m^2},$$

let

$$r(\xi, x) = \sqrt{(x_1-\xi_1)^2 - (x_2-\xi_2)^2 - \cdots - (x_m-\xi_m)^2}$$

denote Lorentzian distance between points $x = (x_1, \ldots, x_m)$ and $\xi = (\xi_1, \ldots, \xi_m)$ in real $m$-dimensional space $E^m$, for which the quantity under the root sign is positive. Let $S$ be an $(m-1)$-dimensional space-like surface, in this case a surface on which $dx_1^2 - dx_2^2 - \cdots - dx_m^2 < 0$ (for instance, the plane $x_1 = 0$). Let $C^\xi$ denote the characteristic cone $\{x \in E^m: r(\xi, x) = 0\}$ with vertex $\xi$, $S^\xi$ the piece cut out of $S$ by this cone, and $D_S^\xi$ the $m$-dimensional conical region enclosed between $S^\xi$ and $C^\xi$. Riesz bases his treatment of Cauchy's problem for the d'Alembertian on the properties of a one-parametric family of integrals

$$I^\alpha v(\xi) = \frac{1}{H(\alpha)} \int_{D_S^\xi} v(x) r(\xi, x)^{\alpha - m} dx,$$

convergent with bounded $v$ for $\alpha > m - 2$. The constant $H(\alpha) = H_m(\alpha)$ is determined in such a way that these integrals, which in structure generalize the classical integrals of Riemann and Liouville, also behave like them. For instance, $I^\alpha(I^\beta) = I^{\alpha+\beta}$, $\square I^{\alpha+2} = I^\alpha$; also,

$$\square\left[r(\xi, x)^{\alpha - m + 2}/H(\alpha + 2)\right] = r(\xi, x)^{\alpha - m}/H(\alpha),$$

a relation used later in Green's formula. In addition, for sufficiently smooth $v$, the function of $\alpha$, $I^\alpha v$, can be analytically and uniquely continued to values of $\alpha$ that are $\leqslant m - 2$, and under this continuation $I^0 v = v$. If $\square v = f$ in $D_S^\xi$, Riesz applies Green's formula over $D_S^\xi$ using $r(\xi, x)^{\alpha - m + 2}/H(\alpha + 2)$ with $\alpha > m - 2$ as auxiliary function. This function vanishing on $C^\xi$, and in view of a previously noted relation, the result can be written as

$$I^\alpha v(\xi) = \frac{1}{H(\alpha + 2)} \int_{D_S^\xi} f(x) r(\xi, x)^{\alpha + 2 - m} d\xi + \frac{1}{H(\alpha + 2)} \int_{S^\xi}$$

$$\left\{ \frac{dv(x)}{dn} r(\xi, x)^{\alpha + 2 - m} - v(x) \frac{d}{dn} r(\xi, x)^{\alpha + 2 - m} \right\} dS.$$

Here, $d/dn$ is Lorentzian "co-normal" differentiation, which on $S$ is differentiation in a certain outward direction, but is in a tangential direction on $C^\xi$; $dS$ is element of area in a Lorentzian sense. Like $I^\alpha v$ in the first member, both integrals in the second member of this relation can be continued analytically to values of $\alpha \leqslant m - 2$. Making the continuation down to the value $\alpha = 0$ provides the representation of $v = I^0 v$ desired.

Riesz extended these methods to equations with variable coefficients by using characteristic conoids in place of characteristic cones and Hadamard's geodetic in place of Lorentzian distance. He constructed auxiliary functions for use in Green's theorem in the general case by means similar to Hadamard's. His analytic continuations originally cost a great deal of effort, but were much simplified in a second article [10] appearing 12 years after the first. With this amendment, his treatment becomes to my mind a model of lucid and inviting mathematical exposition, with which I should urge anyone interested in hyperbolic equations to become familiar.

Hadamard's and Riesz's theories of linear hyperbolic second-order partial differential equations helped give impetus to the development of the theory of distributions with its abstract calculus of fundamental solutions. Friedlander in this work in effect is helping to repay the debt thus owed by distribution theory to Cauchy's problem, reformulating Hadamard's and Riesz's thinking in terms of fundamental solutions and other distributions. This is a substantial achievement carried out in full detail. To sketch the ideas, let $(f, \varphi)$ denote the linear form associated with a given distribution $f$ acting on test functions $\varphi$, which are functions of class $C^\infty(E^m)$ and of compact support. (Sometimes it is useful to write $(f, \varphi)$ with a dummy variable as, say, $(f(x), \varphi(x))$ and also to refer to $f(x)$ and $\varphi(x)$ in place of $f$ and $\varphi$.) The equation $Mu = f$ for distributions $u$ and $f$ is to be interpreted as the totality of relations $(u, M^*\varphi) = (f, \varphi)$ for all test functions $\varphi$. A distribution $u$ satisfying this equation can be constructed if $M$ has a "fundamental solution". This is defined to be a distribution $G_\xi(x)$ depending on a specified point $\xi$, acting on test functions $\varphi(x)$, and satisfying the condition $MG_\xi(x) = \delta(x - \xi)$, which means that $(G_\xi, M^*\varphi) = \varphi(\xi)$ for all test functions $\varphi$. Indeed, if $u$ is the distribution defined by the relations

$$(*) \qquad (u, \varphi) = \big(f(\xi), (G_\xi, \varphi)\big)$$

for all test functions $\varphi$, then the equalities

$$(u, M^*\varphi) = \big(f(\xi), (G_\xi, M^*\varphi)\big) = (f(\xi), \varphi(\xi))$$

show immediately that $Mu = f$. Formula $(*)$ is adapted to Cauchy's problem in the following way. Suppose $u(x)$ to be a smooth solution of $Mu = f$ in a geodesically convex neighborhood $N$ of a space-like surface $S$. Representations of $u(\xi)$ are desired for points $\xi$ on one side of $S$. Denote by $J^+(S)$ the aggregate of all points on this side of $S$ that also lie inside conoids $C^x$ with vertices $x$ on $S$. If $\xi \in J^+(S) \cap N$, Friedlander constructs a fundamental solution $G_\xi^+$ of $M$ supported in $J^+(S)$ and acting on test functions with support in $N$. Then in formula $(*)$ he takes $\chi u$ in place of $u$, where $\chi$ is the characteristic function of $J^+(S)$, to obtain $(\chi u, \varphi) = (M_\xi(\chi u)(\xi), (G_\xi^+, \varphi))$ for all test functions $\varphi$ with support in $N$. (The suffix $\xi$ attached to $M$ signifies that the coefficients in $M$ are to be evaluated at $\xi$ and the differentiations performed with respect to the $\xi^i$.) Since $M(\chi u) = \chi f +$ distributions resulting from the differentiation of $\chi$ and thus supported on the boundary of $J^+(S)$, we have $(\chi u, \varphi) = (\chi(\xi) f(\xi), (G_\xi^+, \varphi)) +$ linear forms in $\varphi$ containing Cauchy data for $u$ on $S$. (No contribution appears referring to parts other than $S$ of the boundary of $J^+(S)$.) From an explicit expression for $(G_\xi^+, \varphi)$, all the linear forms in the second member of this formula are calculated explicitly to reconstruct the representation for $u(\xi)$, $\xi \in J^+(S) \cap N$, of Hadamard and of Riesz. Like these two predecessors, Friedlander uses coordinate-invariant notation, obviously appropriate in such a geometrical approach to Cauchy's problem and, besides, helpful in his treatment of related geometrical topics. These include characteristic manifolds, characteristic initial-value problems, caustics, the propagation of discontinuities, progressing waves, and the approximations of geometrical optics. He also discusses Huygen's Principle, which pertains to the diffusion of waves, in this being necessarily sketchy, but giving certain topics in detail. Although primarily concerned with scalar wave equations, he treats equations of tensor type as well. Most of the book is devoted to four-dimensional space-time ($m = 4$), in which he uses constructions closely parallel to Hadamard's, but in his final chapter he handles all dimensions $m \geqslant 2$ by adapting Riesz's idea of analytic continuation to the construction of fundamental solutions. His first two chapters are compact summaries of results he will need from differential geometry and from the theory of distributions on manifolds, with appropriate references; an appendix covers ideas required from topology. Besides this kind of background, a prospective reader would do well to acquaint himself with the classical theories on which the author's treatment is modeled. Hadamard's method in relation to the simplest case, that of the wave equation with constant coefficients, is explained briefly and with exceptional clarity in the original German edition of Courant-Hilbert, vol. 2 [1, pp. 430–442]. Riesz brings out his own ideas very effectively in a 1960 study of the wave equation [9], as well as in the two works previously mentioned. Also helpful may be the discussions in Courant-Hilbert, vol. 2 [2], of the propagation of discontinuities and progressing waves (pp. 618–642) and of Friedlander's own problem of adapting distribution theory to Hadamard's approach (pp. 740–744).

The author's style is terse and made more difficult by the presence of many misprints. Too numerous to list, these are not likely, however, to mislead the kind of reader to whom this sophisticated monograph is addressed.

No single treatise could do justice to all the approaches to Cauchy's problem that have been invented since 1920. In some of these alternatives to the Hadamard-Riesz-Friedlander direction, characteristic conoids again play essential roles, but much simplicity is gained by using less singular auxiliary functions, which lead to integral equations for solutions of Cauchy's problem. The integral equations are essentially of Volterra's type and can be solved by iteration. The earliest such scheme and one of the most attractive is due to M. Mathisson [7]. S. Sobolev and Y. Choquet-Bruhat have developed a different idea in several papers, which Friedlander references. Other ways to derive integral equations are given by the reviewer [4] and D. Sather [11].

Courant and Lax produce a formula for the solution of Cauchy's problem by relatively simple methods that depend on the theory of discontinuous progressing waves and the theory of spherical means, and without heavy geometrical machinery. Their original treatment [3] is in classical terms, a later account [2, pp. 727–736] in the language of distribution theory.

In the so-called energy methods for Cauchy's problem, characteristic conoids enter unobtrusively, almost invisibly. These methods are based on the energy inequality, which in its usual form refers to a coordinate system in which time $t \equiv x^1$ is distinguished from spatial variables $y \equiv (x^2, \ldots, x^m)$, and in which the plane $t = 0$ is the initial surface. Let $Z = \{(t, y)\colon c\sqrt{\Sigma_{i=2}^m (x^i - a^i)^2} \leq a^1 - t, 0 \leq t \leq T\}$ be a slice of a (solid) cone, with $0 \leq T \leq a^1$, and with small enough positive constant $c$ to make the conical mantle $c\sqrt{\Sigma_{i=2}^m (x^i - a^i)^2} = a^1 - t$ space-like. Let $B_t$ denote the section $\{(t, y)\colon c\sqrt{\Sigma_{i=2}^m (x^i - a^i)^2} \leq a^1 - t\}$ of this cone at height $t$. If $u(t, y)$ is a solution in $Z$ of $Mu = f$, the energy inequality provides a bound for

$$E(t) = \int_{B_t} \left\{ (\partial u/\partial t)^2 + \sum_{i=2}^{m} (\partial u/\partial x^i)^2 \right\} dy$$

in terms of $E(0)$ and an integral containing $f$. It leads to a uniqueness theorem immediately and to an existence theorem in various possible ways. A comprehensive discussion of energy methods, and further references, are given in Courant-Hilbert, vol. 2 [**2**, pp. 652–661, 668–671].

Many important questions about hyperbolic equations are being studied today besides regular linear initial-value problems in space-time regions near a space-like (or characteristic) initial surface, to which our discussion and Friedlander's book for the most part have been confined. Among the other questions are, for instance, boundary-value problems, effects of nonlinearity, the long-range behavior of solutions, and scattering. In the extensive field thus evidenced, Friedlander's monograph is an outstanding instance of unified, detailed treatment of advanced ideas, worthwhile to anyone who makes the preparation called for, indispensable to specialists.

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AVRON DOUGLIS

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