BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 2, March 1978
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Littlewood-Paley and multiplier theory, by R. E. Edwards and G. I. Gaudry, Springer-Verlag, Berlin, Heidelberg, New York, 1977, ix + 212 pp., $25.60.

Two results, obtained in the early thirties, due to Littlewood and Paley [11], can be considered to be the beginning of the Littlewood-Paley theory. Suppose $f \in L^p(T)$, $1 < p < \infty$, where T is the one-dimensional torus, $c_k = (1/2\pi)\int_{-\pi}^{\pi} f(\varphi)e^{-ik\varphi}\,d\varphi$ and $\sum_{k=-\infty}^{\infty} c_k e^{ik\theta}$ is the Fourier series of f. For $N \geqslant 0$ let

$$\Delta_{\pm(N+1)}(\theta) = \Delta_{\pm(N+1)}(\theta; f) = \sum_{2^N \leqslant \pm k < 2^{N+1}} c_k e^{ik\theta} \tag{1}$$

and $\Delta_0(\theta) \equiv c_0$. The first result is that there exist constants A_p and B_p such that

$$A_p\|f\|_p \leqslant \|d(f)\|_p \leqslant B_p\|f\|_p, \tag{2}$$

where $d(f) = (\sum_{N=-\infty}^{\infty}|\Delta_N|^2)^{1/2}$. When $p = 2$, Plancherel's theorem immediately shows that both of these inequalities are equalities with $A_p = 1 = B_p$. When $p \neq 2$ these inequalities give us a characterization of those trigonometric series that are Fourier series of L^p functions (to wit: $\|(\sum_{-\infty}^{\infty}|\Delta_N|^2)^{1/2}\|_p < \infty$). One of the important features of this characterization is that linear operators obtained by multipliers m_k (of the Fourier coefficients c_k) that vary boundedly on the dyadic blocks Δ_N preserve the class L^p. For example the projection of f onto the trigonometric series of power series type, $\sum_{k=0}^{\infty} c_k e^{ik\theta}$, is immediately seen to be bounded on $L^p(T)$ for $1 < p < \infty$. More generally, the famous Marcinkiewicz theorem stating that for $1 < p < \infty$

$$\begin{aligned} &\left\|\sum_{k=-\infty}^{\infty} m_k a_k e^{ik\theta}\right\|_p \\ &\leqslant A_p\left\{\left(\sup_{N\geqslant 0}\sum_{2^N\leqslant|k|<2^{N+1}}|m_{k+1}-m_k|\right)+\sup_k|m_k|\right\}\|f\|_p \end{aligned} \tag{3}$$

is a consequence of (2).

The second result involves the Littlewood-Paley g-function

$$g(f) = \left(\int_0^1 (1-r)|P_r' * f|^2\,dr\right)^{1/2},$$

where P_r' is the derivative (with respect to r) of the Poisson kernel

$$P_r(\theta) = (1-r^2)/(1-2r\cos\theta + r^2).$$

Again we have inequalities which, like (2), express the equivalence of the L^p-norms of f and $g(f)$ provided $\int_{-\pi}^{\pi} f = 0$: for $1 < p < \infty$ there exist constants A_p and B_p such that

$$A_p\|f\|_p \leqslant \|g(f)\|_p \leqslant B_p\|f\|_p. \tag{4}$$

It turns out that in the original work of Littlewood and Paley the operators mapping f into $d(f)$ were studied by using the properties of the g functions (see pages 222–224 of the second volume of Zygmund's book **[27]** for an illuminating heuristic discussion of this connection).

In the paper cited above Littlewood and Paley did show the connection of these results with the boundedness of certain multiplier operators. At about the same time, Paley **[15]** obtained the analog of (2) for another orthonormal system, the Walsh-Paley system; he also obtained multiplier results of the type we have described. There were other papers that can be considered "precursors" of the Littlewood-Paley theory. Among these we cite some of the work of Kaczmarz **[10]** and Zygmund on Cesàro summability. Of particular importance, however, is a function introduced by Lusin **[12]** which is usually referred to as the Lusin area function. Its definition is of a geometric nature and is the following one: if $0 < \delta < 1$, let C_δ denote the circumference $|z| = \delta$ and Ω_δ the open region bounded by the two tangents from $|z| = 1$ to C_δ and by the more distant arc of C_δ between the points of contact. By $\Omega_\delta(\theta)$ we mean the domain Ω_δ rotated through an angle θ around $z = 0$. Now suppose f is a real-valued L^p function with Fourier series $\Sigma_{-\infty}^{\infty} c_k e^{ik\theta}$ and $F(z) = \Sigma_{k=0}^{\infty} c_k z^k$. (Thus, F is analytic in $D = \{z \in \mathbf{C}\colon |z| < 1\}$.) The Lusin area function of f is

$$s(f)(\theta) = s_\delta(f)(\theta) = \left(\int_{\Omega_\delta(\theta)} |F'(z)|^2\, dx\, dy\right)^{1/2}. \tag{5}$$

The reason for its name is that $[s(f)(\theta)]^2$ is the area of the image of $\Omega_\delta(\theta)$ under F ($|F'(z)|^2$ being the Jacobian of the transformation $z \to F(z)$). Lusin showed that f and $s(f)$ have equivalent L^2 norms for each such δ. In complete analogy with (2) and (4) it can be shown that the L^p norms of these two functions are equivalent (when $\int_{-\pi}^{\pi} f = 0$):

$$A_p\|f\|_p \leqslant \|s(f)\|_p \leqslant B_p\|f\|_p \tag{6}$$

for appropriate constants A_p and B_p, $1 < p < \infty$. The second inequality was first established by Marcinkiewicz and Zygmund **[13]**, the first follows from the second and a duality argument (this is often the case for the functions we are discussing). The area function can also be used to obtain multiplier theorems and is closely connected with the two functions $d(f)$ and $g(f)$; for example, it is not hard to show that $g(f) \leqslant A_\delta s_\delta(f)$ for an appropriate constant A_δ.

There are important geometric properties enjoyed by s_δ that stem from the fact that $|F'(z)|$ is the Jacobian of the transformation F. It can be shown (see Marcinkiewicz and Zygmund **[13]** and Spencer **[17]**) that for a function F, analytic in D, the finiteness of the integral in (5) for θ in a set $E \subset [-\pi, \pi)$ is equivalent to the existence of the nontangential limits of $F(z)$ at almost every point of E (by this we mean that z approaches $e^{i\theta}$, $\theta \in E$, within one of the domains $\Omega_\delta(\theta)$). Moreover, the area function and its properties were the key to recently obtained deep understanding of the structure of the Hardy spaces $H^p, p > 0$ (that we shall discuss briefly later on).

These three functions and their uses is the essence of the Littlewood-Paley theory that was developed in the thirties and forties. There are other variants, for example the $g^*(f)$ function and the Marcinkiewicz function $\mu(f)$ (see **[18]**). We shall, however, restrict our attention to the three functions we just introduced as a basis for a short description of the Littlewood-Paley theory and its extensions. Before doing this, however, we would like to point out that the classical theory was developed by making use of "complex methods" (that is, it used results in function theory). A consequence of this was that the theory was one-dimensional. One important development in classical harmonic analysis during the last thirty years has been the extension of classical one-dimensional results to $\mathbf{R}^n$. This followed two main directions: the extension of "real methods" (basically by the techniques introduced by Calderón and Zygmund in their work on singular integrals) and the various extensions of complex methods which were mainly achieved by the introduction of various H^p space theories (see **[19]**, **[8]**, [7]). The corresponding extension of the Littlewood-Paley theory (which bridges the gap between the complex and real method) was mainly carried out by E. M. Stein who developed the theory considerably, widening its applicability both in the classical setting involving $\mathbf{R}^n$ (even when $n = 1$) and in abstract situations involving, among other things, Lie groups, symmetric spaces, diffusion semigroups and martingales (this last setting arose independently from a probabilistic point of view that will be discussed later on). For this reason we feel that it is appropriate to call the modern version of the Littlewood-Paley theory the Littlewood-Paley-Stein (LPS) theory.

In order to understand the more general Littlewood-Paley functions it is useful to express $d(f)$ in terms of a convolution. This can be done by letting

$$\psi_{\pm(N+1)}(\theta) = \frac{1}{2\pi} \sum_{2^N < \pm k < 2^{N+1}} e^{ik\theta}$$

for $N \geqslant 0$ and $\psi_0(\theta) \equiv 1/2\pi$. Then, clearly,

$$d(f) = \left(\sum_{N=-\infty}^{\infty} |\psi_N * f|^2 \right)^{1/2}.$$

The one dimensional situation we have described up to now involves the circle. The functions we have introduced, however, have analogs for the harmonic analysis on $\mathbf{R}$. For example, the g function in this case is given by

$$g(f) = \int_{-\infty}^{\infty} |yP_y' * f|^2 \frac{dy}{y},$$

where P_y' is the derivative (with respect to y) of the Poisson kernel associated with the real line

$$P_y(x) = \frac{y}{x^2 + y^2} = \frac{1}{y} \psi\left(\frac{x}{y}\right)$$

(here $\psi(x) = 1/(1 + x^2)$).

The basic Littlewood-Paley function (or operator) on $\mathbf{R}^n$ is obtained by considering an integrable function ψ such that $\int_{\mathbf{R}^n} \psi(x)\, dx = 0$ and writing

$$g_\psi(f) = \left(\int_0^{\infty} |\psi_\varepsilon * f|^2 \frac{d\varepsilon}{\varepsilon} \right)^{1/2}$$

where $\psi_\varepsilon(x) = (1/\varepsilon^n)\psi(x/\varepsilon)$. In certain important cases g_ψ can be defined when ψ is a distribution. A more elementary version of a Littlewood-Paley function is the following

$$d_\psi(f) = \left(\sum_{k=-\infty}^{\infty} |\psi_{2^k} * f|^2 \right)^{1/2}.$$

All the functions we have introduced are of the form

$$g_k(f) = \|k * f\|_H,$$

where k is a kernel with values in a Hilbert space H and $\| \ \|_H$ is the norm on H. For the three classical operators d, g and s the associated Hilbert spaces are l^2, $L^2(0, 1)$ and $L^2(\Omega_\delta(0))$ (with appropriate measures on $[0, 1]$ and $\Omega_\delta(0)$). The fact that a Hilbert space is involved in this manner in the definition of these functions is one of their important features. For example, the duality argument alluded to, following inequalities (6), is particularly natural because of this. Moreover, in many cases these "g-functions" (or "g-operators") can be viewed as vector-valued Calderón-Zygmund singular integrals. A basic property of these functions, of course, is the equivalence of the L^p norms of f and $g_k(f)$:

$$A\|f\|_p \leqslant \|g_k(f)\|_p \leqslant B\|f\|_p, \tag{7}$$

where A, B and p depend on k. When $p = 2$ these inequalities are (usually) particularly easy to prove by making use of the Plancherel theorem or some other Hilbert space techniques. When p is close to 1 (or ∞) the inequalities are often obtained by employing some version of the Calderón-Zygmund methods. Sharp results for p close to 2 are frequently obtained by embedding k into an analytic family, k_z, of kernels and using the estimates we just mentioned (at 2 and near 1) in conjunction with Stein's interpolation theorem for analytic families of operators. The power of the LPS theory stems from the fact that this program is quite general and can be carried out when all other methods seem to fail.

These operators are obviously of a technical nature. Their use is in much the same spirit as the use that is made of the Hardy-Littlewood maximal operator to control the size of certain basic convolution operators. More specifically, suppose $f \in L^p(\mathbf{R}^n)$ and $Mf = (m\hat{f})^\vee$; a technique developed by Stein for proving that M is bounded on $L^p(\mathbf{R}^n)$ is to show that a pointwise inequality of the form

$$g_2(Mf) \leqslant g_1(f) \tag{8}$$

holds for two appropriate Littlewood-Paley functions g_1 and g_2 (and yet f and Mf are in no way pointwise comparable). The boundedness of M is then an immediate consequence of (7) and (8). A simple example of this situation occurring when $n = 1$ is given by the Hilbert transform: Here we can take $m(x) = \operatorname{sgn} x$ and it follows easily that $s(Mf) = s(f)$.

The following example illustrates the program we mentioned above and shows how the LPS theory involves both the real and complex methods. Stein recently obtained the following remarkable real variable result (see [23]):

Suppose Σ_{n-1} is the surface of the unit sphere in $\mathbf{R}^n$ and σ the Lebesgue surface measure of Σ_{n-1}. We can define

(9) $$(M_r f)(x) = \int_{\Sigma_{n-1}} f(x - ry)\, d\sigma(y)$$

for $f \in L^p(\mathbf{R}^n)$ sufficiently smooth. Let $(\mathfrak{M}f)(x) = \sup_r |(M_r f)(x)|$. Stein's result is that if $n/(n-1) < p \leqslant \infty$, $n \geqslant 3$, then $\mathfrak{M}$ is a bounded operator on L^p. When $n \geqslant 2$, the result is false for $p \leqslant n/(n-1)$; it is an open question whether the positive result is valid when $n = 2$.

By introducing the means $(M_r^\alpha f)(x) = (f * m_{\alpha,r})(x)$, where $m_\alpha(x) = (1 - |x|^2)^{\alpha-1}/\Gamma(\alpha)$ for $\alpha > 0$ and $m_{\alpha,r}(x) = r^{-n} m_\alpha(x/r)$, we obtain the means in (9) as the limiting case as $\alpha \to 0$. By taking Fourier transforms we see that we can embed all these means in an analytic family (where α is complex). One then estimates the L^2 norms of the "g-function"

$$g_\alpha(f)(x) = \left(\int_0^\infty |(M_r^\alpha f)(x) - (f * \varphi_r)(x)|^2 \frac{dr}{r} \right)^{1/2},$$

where $\alpha > 1/2 - n/2$ and φ is a smooth function such that $\hat{\varphi}(0) = \hat{m}_\alpha(0)$. Stein's result is then obtained from the interpolation theorem we mentioned above and easier inequalities for p near 1 and $p = \infty$.

We illustrate the connection between maximal- and g-functions in a simpler situation that uses the discrete version of the g function in question. This will yield a variant of Stein's result for which the same kind of estimates hold for the maximal function

(10) $$\sup_{k \in \mathbf{Z}} |(M_{2^k} f)(x)| \equiv m_f(x).$$

It is easier to explain the technique involved in this case. Briefly, and more generally, the idea is as follows:

Let $d\varphi$ and $d\theta$ be the elements of two measures on $\mathbf{R}^n$ such that $\int_{\mathbf{R}^n} d\varphi = 1 = \int_{\mathbf{R}^n} d\theta$. Let

$$(\varphi^* f)(x) = \sup_{j \in \mathbf{Z}} \left| \int_{\mathbf{R}^n} f(x - 2^j y)\, d\varphi(y) \right|$$

and $(\theta^* f)(x)$ be defined similarly. Assume θ^* is bounded on L^p. To obtain similar information for φ^* one considers the Littlewood-Paley function

$$\sigma(f) = \left[\sum_{j=-\infty}^{\infty} \left(\int_{\mathbf{R}^n} f(x - 2^j y)\, d[\theta(y) - \varphi(y)] \right)^2 \right]^{1/2}$$

then $\varphi^* \leqslant \theta^* + |\varphi - \theta|^*$ and $|\varphi - \theta|^*(f) \leqslant \sigma(f)$. The L^2 estimate now follows from the Plancherel theorem provided

$$\sum_{-\infty}^{\infty} \left| \hat{\theta}(2^j y) - \hat{\varphi}(2^j y) \right|^2 \leqslant c.$$

In the situation we described in order to state Stein's result this inequality is easily established. In order to obtain L^p estimates one can follow the program we outlined above. It is interesting to note that for the maximal function defined by (10) we do obtain positive results when $n = 2$; in fact it can be shown that $\|mf\|_p \leqslant A_p \|f\|_p$ for $1 < p < \infty$ and all dimensions $n \geqslant 2$.

Another important extension of the classical Littlewood-Paley theory is to

H^p spaces. Calderón [5] characterized the functions $F \in H^p$ as those for which $\|s(F)\|_p < \infty$. He first used this result in his paper on commutator singular integrals [11]. Recently he employed it in his solution of the famous conjecture about the existence of the Cauchy integral on curves. More precisely, he showed that if f is an integrable function on a rectifiable curve Γ then the limits

$$\lim_{z \to z_0} \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

exist a.e. as long as z approaches $z_0 \in \Gamma$ nontangentially [6]. Stein ([20] and [21]) extended the Littlewood-Paley theory on H^p spaces and obtained a variant of the Marcinkiewicz theorem. A major advance in this direction was the characterization of all the classical H^p spaces, $0 < p \leqslant 1$, in terms of certain maximal functions (see Burkholder, Gundy and Silverstein [3]) where the s-function played an important role. Fefferman and Stein [8] then extended this characterization to n-dimensions; in their treatment the n-dimensional s-function played a crucial role.

As we mentioned above there are several other settings in which the LPS-theory applies and Stein has been responsible for most of this development. Quite a bit of this work is treated in his book *Topics in harmonic analysis related to the Littlewood-Paley theory* (for a complete reference see [22]). There he develops the theory for compact Lie groups and shows how to obtain a central multiplier theorem. He also deals with symmetric diffusion semigroups; a family $\{T^t\}$, $0 \leqslant t \leqslant \infty$, of operators mapping functions on some measure space to functions on the same space such that $T^{t_1+t_2} = T^{t_1}T^{t_2}$, T^0 = identity and

(I) (Contraction property) $\|T^tf\|_p \leqslant \|f\|_p$, $1 \leqslant p \leqslant \infty$;
(II) (Symmetry property) T^t is selfadjoint on L^2, for each $t > 0$;
(III) (Positivity property) $T^tf \geqslant 0$ if $f \geqslant 0$;
(IV) (Conservation property) $T^t 1 = 1$.

The Poisson integral in $\mathbf{R}^n$ ($T^tf = P_t * f$) is an example. Such semigroups occur often in analysis, particularly in connection with differential equations. Stein introduces the g-function

$$g(f) = \left(\int_0^\infty \left| t \frac{\partial T^t}{\partial t} f \right|^2 \frac{dt}{t} \right)^{1/2}$$

and obtains L^p estimates for this operator as well as the maximal operator

$$f^*(x) = \sup_{t>0} |(T^tf)(x)|.$$

He establishes the following multiplier theorem: Let $T^t = \int_0^\infty e^{-\lambda t}\, dE(\lambda)$ be the spectral representation for T^t and $m(\lambda) = \lambda \int_0^\infty b(t) e^{-\lambda t}\, dt$, $\lambda > 0$, for some $b \in L^\infty$, then

$$f \to \int_0^\infty m(\lambda) dE(\lambda) f$$

is bounded on L^p, $1 < p < \infty$.

These results depend on the Littlewood-Paley theory for martingales (that we shall describe briefly in a moment). The latter can be linked to certain discrete semigroups (by a theorem of Rota [**16**]). The results are then sharpened considerably by Stein by extending these semigroups to analytic semigroups and using ingenious complex interpolation arguments.

Stein's book ends with a chapter devoted to several other applications to LPS theory and suggestions for further research. Though this book has been out for seven years many of these suggestions are still worthwhile to follow.

We already mentioned that there has been an independent development of the Littlewood-Paley theory motivated by a probabilistic point of view. In some sense it is fair to say that the beginning of this approach is indicated by the early paper of Paley involving Walsh-Paley series (which we have already cited). The modern theory, however, was started by Burkholder [**2**] (see also Austin [**1**]). Very briefly, in the martingale setting the relevant Littlewood-Paley function is what is often called the "square function", which is defined for a martingale $f = \{f_n\}$ to be

$$s(f) = \left(\sum_i |f_i - f_{i+1}|^2\right)^{1/2}$$

(this is the correct definition when the underlying measure space has infinite measure; otherwise the square of the mean of f has to be added). A typical class of theorems involve relations between $s(f)$ and $f^* = \sup_i|f_i|$. In particular, the fact that $\|s(f)\|_p$ is equivalent to $\|f^*\|_p$ for $0 < p < \infty$, which was first proved in the probabilistic setting, led to the characterization of H^p, $p > 0$, by Burkholder, Gundy and Silverstein (which we mentioned above). We have used the same notation for the area function and the square function. We will not go into it any further here, but we would like to mention that there are natural interpretations of the classical g-function and the s-function in the martingale setting. It turns out that these interpretations lead one to the conclusion that the square function is the appropriate analog of these two classical functions. A complete discussion of these matters is in P. A. Meyer's presentation in [**14**, pp. 125–161]. We refer the reader to the book by Taibleson [**25**] for a treatment of the Littlewood-Paley theory that connects the probabilistic and harmonic analysis point of view in the setting of local field theory. An excellent treatment of martingales can be found in Garsia's book [**9**].

We have by no means exhausted all the applications of the LPS theory, nor have we described all of its features. For example, the various inclusion relationships between the various Lipschitz spaces (or Potential spaces) were obtained by Taibleson [**24**] (also see p. 155 of [**21**]), by using LPS theory. There are many other applications; among these, some can be found in the book being reviewed. We hope, however, that this description gives some of the flavor and imparts the importance of the LPS theory.

Edwards and Gaudry in their book, which is being reviewed, concentrate mainly on multiplier theory on some locally compact abelian groups. They present a detailed account of those aspects of the Littlewood-Paley theory which generalizes the classical d-function and its relation to multiplier theorems of the Marcinkiewicz type. Their book treats separately **R**, **Z**, **T**, certain disconnected groups and martingales in the first seven chapters. One of the main objectives of the authors is to present a complete proof of the Marcinkiewicz multiplier theorem for the above mentioned groups. This is done in the eighth chapter. The last chapter contains applications to Hadamard and Sidon sets and the existence of a "singular" multiplier is also established.

This book does serve as a self-contained introduction to those aspects of the Littlewood-Paley theory which limit themselves to the function d and to those methods that are based on the Calderón-Zygmund theory. We feel, however, that the reader should be warned that one of the main features of the LPS-theory, the fact that it bridges the gap between real and complex methods, plays no part in this book. The authors state "that by no means all aspects of Littlewood-Paley theory or of multiplier theory are dealt with" in their book. Consequently we would not ordinarily fault them for not mentioning anything about a main feature of the theory; nevertheless, we do take exception to the comments that are made about the g-functions: "these are quadratic functionals involving derivatives of the Poisson integral of f, arising through connections with analytic and harmonic functions. As such, they are somewhat more remotely connected with pure harmonic analysis and we shall leave them aside." As we have seen, the g functions and their analogs play a central role in some of the more important applications of the theory. In our opinion, the connection between the various other Littlewood-Paley functions and their applicability to many aspects of harmonic analysis should be stressed and not ignored.

References

1. D. G. Austin, *A sample property of martingales*, Ann. Math. Statist. **37** (1966), 1396–1397.

2. D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist. **37** (1966), 1494–1504.

3. D. L. Burkholder, R. F. Gundy and M. L. Silverstein, *A maximal function characterization of the class H^p*, Trans. Amer. Math. Soc. **157** (1971), 137–153.

4. A. P. Calderón, *Commutators of singular integral operators*, Amer. J. Math. **78** (1956), 310–320.

5. ———, *Algebras of singular integral operators*, Proc. Sympos. Pure Math. vol. 10, Amer. Math. Soc., Providence, R.I., 1967, pp. 18–55.

6. ———, *Cauchy integrals on Lipschitz curves and related operators*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), 1324–1327.

7. R. R. Coifman and Guido Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.

8. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

9. A. M. Garsia, *Martingale inequalities*, Benjamin, New York, 1973.

10. S. Kaczmarz, *Über die Konvergenz der Reihen von Orthogonalfunktionen*, Math. Z. **23** (1925), 263–270.

11. J. E. Littlewood and R. E. A. C. Paley, *Theorems on Fourier series and power series*, J. London Math. Soc. **6** (1931), 230–233.

12. N. Lusin, *Sur une propriété des fonctions à carré sommable*, Bull. Calcutta Math. Soc. **20** (1930), 139–154.

13. J. Marcinkiewicz and A. Zygmund, *A theorem of Lusin*, Duke Math. J. **4** (1938), 473–485.

14. P.-A. Meyer, *Démonstration probabiliste de certains inégalités de Littlewood-Paley*, Exposés I and II, Lecture Notes in Math., vol. 511, Springer-Verlag, Berlin and New York, 1976, pp. 125–161.

15. R. E. A. C. Paley, *A remarkable series of orthogonal functions*. I, Proc. London Math. Soc. **34** (1932), 241–264.

16. G. C. Rota, *An "Alternieven de Verfahren" for general positive operators*, Bull. Amer. Math. Soc. **68** (1962), 95–102.
17. D. Spencer, *A function theoretic identity*, Amer. J. Math. **65** (1943), 147–160.
18. E. M. Stein, *On the functions of Littlewood-Paley, Lusin and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466.
19. E. M. Stein and Guido Weiss, *On the theory of harmonic functions of several variables*, Acta Math. **103** (1960), 25–62.
20. E. M. Stein, *Classes H^p, multiplicateurs et fonctions de Littlewood-Paley*, C. R. Acad. Sci. Paris **263** (1966), 716–719.
21. ______, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1970.
22. ______, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Ann. Math. Studies, no. 63, Princeton Univ. Press, Princeton, N.J., 1970.
23. ______, *Maximal functions: Spherical means*, Proc. Nat. Acad. Sci. U.S.A. **73** (1976), 2174–2175.
24. M. H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-spaces*. I, J. Appl. Math. Mech. **13** (1964), 407–480; II (ibid) **14** (1965), 821–840.
25. ______, *Fourier analysis on local fields*, Math. Notes, no. 15, Princeton Univ. Press, Princeton, N.J., 1975.
26. A. Zygmund, *Une remarque sur un théorème de M. Kaczmarz*, Math. Z. **25** (1926), 297–298.
27. ______, *Trigonometric series*, Cambridge Univ. Press, New York, 1959.

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BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 2, March 1978
© American Mathematical Society 1978

Trigonometrical sums in number theory, by I. M. Vinogradov, Statistical Publishing Society, first published: October 1975, Calcutta, viii + 151 pp., \$15.00.

1. Introduction. The theory of trigonometric (sometimes called exponential) sums is so intimately associated with I. M. Vinogradov that a book by this master is a noteworthy event.

The book under review is a translation from the Russian edition of 1970 which, in its turn, is described as a revised edition of Vinogradov's 1947 book of the same title. "Revised edition", however, is a misnomer, since a comparison with the 1947 edition shows that the present work is a complete rewriting and incorporates new refinements and improvements–results mostly due to the author himself.

Trigonometric sums have been used in one form or another in number theory since Gauss's solution of the cyclotomic equation in which he introduced "Gaussian" sums. These led to a highly interesting and somewhat unexpected proof of the quadratic law of reciprocity. There are many other examples and applications. The applications given in this book deal with the Waring problem, the distribution of the fractional parts of polynomials and with the Waring-Goldbach problem. Reference is also made to the applications to the zeta function.

2. The problem of trigonometric sums. The integers are naturally embedded in the complex numbers but a useful point of view for number theory-