Krein, and the technicalities are more involved, but the general features of the solutions are similar to those for Krein's problem.

The reviewer is in the uncomfortable position of not being an expert in prediction theory, the main topic of the book under review. Rather, I am someone who was brought up in Hardy spaces and developed a curiosity about how they get involved with prediction theory. For such a person the book is almost ideal. I imagine the same would be true for someone reared in probability theory who developed the complementary curiosity to mine. The book begins with three short but intense preparatory chapters which provide the needed background in function theory, Hardy spaces, and probability. The fourth chapter deals with various prediction problems, beginning with the Kolmogorov-Wiener problem mentioned above. The central theme is an effort to express in terms of the spectral measure Δ the amount of dependence between the past and the future of the process. In the two remaining chapters, Krein's theory of strings and its connection with de Branges spaces of entire functions are developed in detail and applied in the manner sketched above.

I found the comparatively informal style of the book congenial and effective. Many details of proofs are left to the reader in the form of carefully prepared exercises. The authors have clearly made an effort to write a book that will be of value to the learner. If my experience is typical, they have succeeded.

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Probability methods for approximations in stochastic control and for elliptic equations, by Harold J. Kushner, Academic Press, New York, San Francisco, London, 1977, xvii + 243 pp. $23.00.

The analysis of the transition from Markov chains to diffusions, the convergence of solutions of difference equations to corresponding ones for differential equations and related approximation problems have been studied intensively for many years and appear frequently in so many different specialized contexts that it is practically impossible today to have a comprehensive idea of what goes on in the field. Kushner's work aims directly at a specific class of approximations for optimal diffusion processes which are associated with partial differential equations (PDE). In this way he limits the material to manageable size which one can divide, roughly, into two parts.

The first one is the content of Chapters one to seven and Chapter ten and deals with background material, the theory of weak convergence of measures (without details), and the convergence of (nonoptimal) chains to diffusions. The second part, the main point of the book, is the content of Chapters eight and nine and deals with the approximation of optimal diffusions. Chapter eleven deals with a special topic, the separation theorem of stochastic control.

Let us look into part one in some detail. The beginning of the theory of approximations of Markov chains by diffusions is probably the well-known work of Khinchine [1]. The analysis here is simple and direct. It is based on the assumption that the limiting diffusion equation has smooth solutions and the problem is reduced to a two term Taylor expansion. This method can easily yield error estimates that depend upon the smoothness of the solution of the diffusion equation. The major drawback of this approach is precisely its reliance on a priori regularity assumptions which are either hard to obtain or nonexistent for the nonlinear problems associated with optimal diffusions. For diffusion in bounded domains Khinchine's method is more involved because he assumes only interior regularity for the PDE and convergence is only proved in the interior (behavior near the boundary is not analyzed).

It was known as early as in 1928 [2] that one could obtain convergence results like Khinchine's *indirectly* by compactness arguments which do not require a priori regularity. In fact, these methods were devised to provide an existence theory for the PDE's under consideration. From the probabilistic point of view the PDE methods of [2], energy identities, compactness, the (precursor of) Sobolev's inequalities, etc., appear somewhat remote and unnatural to the problem, however.

As a very simple example of Khinchine's method consider the quantity

$$u^h(t, x) = E_x\left\{f\big(x^h(t)\big)\right\}$$

which is the expectation of a smooth function of the state $x^h(t)$, with $x^h(0) = x$, of a one dimensional random walk. The random walk moves to the right or to the left a distance h with probability $\frac{1}{2}$ and the time intervals between jumps are independent and exponentially distributed with parameter h^{-2}. It follows that $u^h(t, x)$ satisfies Kolmogorov's equations

$$(1)\quad \frac{du^h(t,x)}{dt} = \frac{1}{h^2}\left(\frac{1}{2}u^h(t, x+h) + \frac{1}{2}u^h(t, x-h)\right) - \frac{1}{h^2}u^h(t,x), \qquad t > 0,$$

$$u^h(0, x) = f(x), \qquad x \in R^1.$$

Now suppose that $u(t, x)$ satisfies the heat equation

$$(2)\qquad \frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\,\frac{\partial^2 u(t,x)}{\partial x^2}, \; t > 0, \qquad u(0, x) = f(x),$$

and that $f(x)$ is smooth so that $u(t, x)$ is a smooth solution. We note that

$$u^h(t,x) - u(t,x) = E_x\left\{u\big(t-s, x^h(x)\big)\Big|_0^t\right\}$$

$$(3)\qquad = E_x\int_0^t\Big[-u_t\big(t-s, x^h(s)\big) - \frac{1}{h^2}u\big(t-s, x^h(s)\big) + \frac{1}{2h^2}u\big(t-s, x^h(s)+h\big) + \frac{1}{2h^2}u\big(t-s, x^h(s)-h\big)\Big]\,ds.$$

This identity is easily verified. From the analytical point of view it is simply an expression for the "error" $u^h - u$ as an integral relative to the Green's function of the finite difference problem (1). Since $u(t, x)$ satisfies the heat equation, the integrand in (3) goes to zero as $h \to 0$ and this shows that indeed $u^h \to u$ as $h \to 0$ for each t and x and uniformly on compact sets.

The smoothness of u plays an important role here. The approximation of (1) by the heat equation is the main objective.

To illustrate the Hilbert space method very simply we describe the method of [2] which contains the essential ideas (which are much more streamlined now). First suppose that we have a solution of (2) over some interval $[a, b] \subset R^1$ with $u(t, a) = u(t, b) = 0$. Multiplying (2) by u, integrating over $[a, b]$ and integrating by parts we get the identity

$$\int_a^b u^2(t, x)\, dx + \int_0^t \int_a^b u_x^2(s, x)\, dx\, ds = \int_a^b f^2(x)\, dx, \tag{4}$$

which is called the energy identity associated with (2). In [2] an identity similar to (4) is obtained for the finite difference equation (1). Actually, the heat equation is not explicitly treated in [2] but the methods carry over. Using this identity they establish a priori estimates independent of h for the L^2 norm of u^h (over the grid contained in $[a, b]$) and the L^2 norm of differences of u^h. From this they deduce equicontinuity of the functions u^h and similarly for finite differences of u^h. Passing to the limit $h \downarrow 0$ they obtain the desired convergence to the solution of (2).

The main point of the approach of [2] is that to pass from (1) to (2) one does not have to know anything about (2) to begin with except uniqueness, which in case of (2) follows also from (4).

The probabilistically natural *indirect* methods first appeared in 1956 [3], [4], after the theory of weak convergence of measures on complete separable metric spaces was developed and after the theory of Markov processes had progressed far enough to provide probabilistic representations for solutions of many PDE problems. In [5] and [6] there are a number of theorems on the convergence of chains to diffusions using indirect, compactness methods.

The best results on the convergence of chains to diffusions were obtained by Stroock and Varadhan [7], [8] who first gave a new formulation of diffusion theory in terms of martingale problems requiring minimal assumptions about the smoothness of the coefficients of PDE. At the same time they introduced a wealth of useful techniques that go much beyond the immediate needs in their papers and the mere removal of regularity assumptions on the coefficients.

Briefly, the martingale formulation works as follows. Let $x^h(t)$, $t > 0$ be the process under consideration whose behavior as $h \downarrow 0$ is to be analyzed. Suppose (for simplicity) that by process we mean a probability measure P^h on $C([0, \infty), R^1) = \Omega$ with the topology of uniform convergence on compact sets. Suppose further that this measure is characterized by the property that a certain class of functionals on Ω are P^h martingales relative to $\mathfrak{F}_t$ = σ-algebra of Borel sets in Ω that involve trajectories up to time t. This corresponds analytically to saying that something is a weak solution of an equation, the martingale characterization being a natural notion of weak solution in the probabilistic context.

For the convergence argument one first shows that the P^h are weakly compact. There are many convenient criteria for this ([3]–[6]); this step is analogous to the step of getting a priori estimates for finite differences in [2]. Next one passes to the limit in the martingale identity characterizing P^h

(using suitable functionals if necessary) along a convergent subsequence. Finally, by independent means one argues that the limit measure P is uniquely characterized by the martingale formulation and hence every sequence of P^h converges as $h\downarrow 0$ to P (this last step may be a difficult one, in general, but not in the heat equation example).

Kushner gives a very clear and well-organized account of the work originating in [3]–[8] along with the necessary background material. This is the content of what we called part one. Among the relatively novel points in his treatment is the good use of the Skorohod representation [5, p. 9] and the device of enlarging the state space to include secondary processes along with the primary ones. This avoids difficulties when the secondary processes are not continuous functionals of the primary one. On the other hand it introduces new difficulties in the identification of the limit. The new problems are not, however, serious and can be easily overcome in the context of *nonoptimal* diffusions, i.e., the linear problems.

Part two (Chapters eight and nine) is the main part of Kushner's work. It treats the convergence of optimal Markov chain approximations of optimally stopped and/or optimally controlled diffusion processes. The discrete approximations are the rather obvious ones where one replaces derivatives by finite differences so that the resulting problem does correspond, consistently with the PDE, to a Markov chain problem. This puts restrictions on the coefficients of the PDE and/or the mesh size that correspond to the usual stability conditions but may be control dependent and quite complicated. In the convergence proof the main difficulty is in identifying properly the limit as an optimal diffusion. This is done very nicely and should be of substantial interest to specialists in stochastic control.

It should be kept in mind that the underlying PDE problem for the optimal diffusions is a nonlinear problem (not at all close, in any sense, to a linear problem). The probabilistic methods that are used in the convergence proof of the optimal chains provide an *existence theory* for the optimal diffusions. Kushner does not mention this. Instead he presents his work as an approximation theory that is aimed at actual computations but does point out that much work needs to be done on the computational aspects. This is definitely an understatement of the situation, especially since the indirect methods of proof that are so attractive and allow such generality give no idea whatsoever about the relative merits of different approximation schemes or other approximation methods in general.

In the reviewer's opinion Kushner's work is an important contribution to the literature of stochastic control. It is particularly useful because the author has organized very well and has presented very clearly the material of the first seven chapters that lead up to the optimal diffusion problems.

References

1. A. Khinchine, *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, Springer, Berlin, 1933.

2. R. Courant, K. Friedrichs and H. Lewy, *Über die partiellen Differentialgleichungen der mathematischen Physik*, Math. Ann. **100** (1928), 32–74. See also the translation, *On the partial difference equations of mathematical physics*, IBM J. Res. Develop. **2** (1967).

3. Yu. V. Prohorov, *Convergence of random processes and limit theorems in probability theory*, Theor. Probability Appl. **1** (1956), 187–214.

4. A. V. Skorohod, *Limit theorems for stochastic processes*, Theor. Probability Appl. 1 (1956), 261–290.

5. ______, *Studies in the theory of random processes*, Addison-Wesley, Reading, Mass., 1965.

6. I. I. Gihman and A. V. Skorohod, *Introduction to the theory of random processes*, Saunders, Philadelphia, Pa., 1969.

7. D. W. Stroock and S. R. S. Varadhan, *Diffusion processes with continuous coefficients*. I, II, Comm. Pure Appl. Math. 22 (1969), 345–400, 479–530.

8. ______, *Diffusion processes with boundary conditions*, Comm. Pure Appl. Math. 24 (1971), 147–225.

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Estimates for the $\bar{\partial}$-Neumann problem, by P. C. Greiner and E. M. Stein, Princeton Univ. Press, Princeton, N. J., 1977, 194 pp., $6.00.

Several complex variables has enjoyed a renaissance in the past twenty-five years, reaching deeply into modern algebra, topology, and analysis for techniques to attack long standing problems. An important example is the question of identifying domains of holomorphy, i.e. those open sets in $\mathbf{C}^{n+1}$ (or, more generally, in complex manifolds) for which at least one holomorphic function has no extension outside the set. Early in this century E. E. Levi defined a condition, now called *pseudoconvexity*, which he proved was necessary, and conjectured was sufficient, to characterise domains of holomorphy. More precisely, for domains with smooth boundary one can define a Hermitian form, now called the *Levi form*, on the space of holomorphic vectors tangent to the boundary. The domain is then called *pseudoconvex* (resp. *strictly pseudoconvex*) if the Levi form is positive semidefinite (resp. definite).

Levi's conjecture for $\mathbf{C}^{n+1}$ was finally proved nearly fifty years later by Oka [16] (and simultaneously by Bremermann, [1] and Norguet [15]) after a long series of related papers. Efforts to extend the results to complex manifolds led Grauert [5] to discover a new, more general proof making extensive use of sheaf theory. A totally different proof was later obtained by Kohn [11] (using a crucial estimate of Morrey [13]) as a consequence of his solution of the "$\bar{\partial}$-Neumann" boundary value problem in partial differential equations.

Since Kohn's breakthrough on the problem there has been considerable interest in constructing solutions for the inhomogeneous Cauchy-Riemann (C-R) equations in a bounded complex domain and studying their boundary behavior. Kohn's methods, based on a priori L^2 estimates, give only L^2 existence proofs for solutions of the C-R equations. (After Kohn's work appeared Hörmander [9] gave a simpler existence proof, using weighted L^2 estimates, in which boundary problems are completely circumvented!) Several explicit solutions have been constructed by the use of integral formulas, in particular, those of Henkin [8] and Ramirez [18]. Kerzman [10], Grauert and Lieb [6], Overlid [17], and others have obtained estimates for these solutions in terms of L^p and Lipschitz norms.

Recently Greiner and Stein were able to give an explicit construction of Kohn's solution and to obtain from this construction optimal estimates in L^p and other norms. The book under review is an exposition of this work,