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Near-rings, the theory and its applications, by Gunter Pilz, Mathematics Studies, No. 23, North-Holland, Amsterdam, 1977, xiv + 393 pp., $24.00.

It has been said that the supreme occurrence in the course of an idea is that brief moment between the time it is heresy and the time it is trite. At the start of the 1960s "nonlinear functional analysis" seemed to strike most mathematicians as a contradiction in terms but by the end of that decade, some functional analysts were apologizing for considering "only the linear case."

During this period many began to realize (or to rediscover from earlier times) that quite a number of pressing scientific problems are nonlinear in nature. At the same time many others began to realize (or again, to rediscover) that many nonlinear problems have a vigorous algebraic life.

It is widely understood that many linear problems have a natural setting in some ring of linear transformations. For two illustrations (out of a vast number of possibilities) consider the following:

(1) A study of the spectrum of a bounded selfadjoint operator T on a Hilbert space H leads naturally to a consideration of the smallest closed subring of $L(H, H)$ which contains T.

(2) A strongly continuous one-parameter semigroup of bounded linear transformations on a Banach space X may be considered as a kind of ray in the ring $L(X, X)$. By now a substantial start for both nonlinear spectral

theory and for one-parameter semigroups of nonlinear transformations has been made but a setting for such developments (parallel to that provided by rings of linear operators for linear problems) has not been clearly established.

To look for such a setting it is natural to turn to the subject of near-rings: A near-ring is a set N with an addition and a multiplication so that N is a group (not necessarily abelian) under addition and is a semigroup under multiplication so that right distributivity holds, i.e.,

$$(x + y)z = xz + yz, \qquad x, y, z \in N.$$

An example of a near-ring is the set of continuous real-valued functions on all of R with addition defined pointwise and multiplication taken to be composition. Clearly left distributivity does not hold here so that one certainly does not have a ring.

In a sense, near-rings provide a nonlinear analogue to the developments of linear algebra. Certainly linear algebra is a subject of vast applicability but it also has great importance as a source of ideas for linear functional analysis. One is reminded that linear functional analysis grew out of linear algebra in response to the needs of linear differential equations. It is likely that knowledge of near-rings together with a good assessment of the needs of modern nonlinear differential equations will lead to fruitful developments in nonlinear functional analysis such as a "nonlinear Gelfand theory" or a much more extensive nonlinear spectral theory. Near-rings of Lipschitz, zero-fixing transformations from a Banach space to itself seem to be potentially valuable for applications.

In passing from linear to nonlinear problems the matter of notation seems particularly important. Linear analysis has long possessed a good notation. For a linear transformation T on a vector space no one speaks of the "linear transformation $T(x)$" where x is supposed to be a "variable" vector. Contrast this with the curse of "the function $f(x)$" which is still being peddled by almost all calculus books. The wide-spread persistence of this deficient functional notation seems a hindrance to general recognition of the underlying function-algebraic aspects of many problems.

This book contains a nearly encyclopedic account of the present state of the algebraic theory of near-rings. It is nearly self-contained for one familiar with ring theory.

To gain some initial insight into near-rings consider the relevant notion of ideal. For a near-ring N, a normal subgroup I of $(N, +)$ is an ideal in N if (α) $IN \subseteq I$ and (β) $n(n' + I) - nn' \in I$, $n, n' \in N$. If (α) is satisfied then I is a right ideal and if (β) is satisfied then I is a left ideal. An ideal turns out to be a homomorphic image of N. The lack of symmetry between left and right ideals is typical for the subject.

Closely connected with near-rings are N-groups. For a near-ring N and group $(\Gamma, +)$, $\mu: N \times \Gamma \to \Gamma$ gives an N-group ($n\gamma \equiv \mu(n, \gamma)$, $\gamma \in \Gamma$, $n \in N$) provided that

$$(n + n')\gamma = n\gamma + n'\gamma, \qquad (nn')\gamma = n(n'\gamma), \qquad n, n' \in N, \gamma \in \Gamma.$$

Such an N-group is denoted by ${}_N\Gamma$.

Various decomposition theorems for near-rings and N-groups are given under assumptions of various chain conditions. As in the whole subject, ring theory gives always at least a rough idea of what to expect.

We describe here some of the ideas which indicate some structure theory for near-rings. A better description undoubtedly could be done by a card-carrying algebraist but here it is anyway.

An N-group $_N\Gamma$ is simple if and only if it has no nontrivial ideals and it is called N-simple if and only if it has no N-subgroups except Ω and Γ ($\Omega \equiv \{n0\}$, $n \in N$). Simple N-groups $_N\Gamma$ such that $N^\Gamma \neq \{0\}$ do not in general have the property that $N\gamma = \{0\}$ or $N\gamma = \Gamma$ for all $\gamma \in \Gamma$ (unlike the ring-module case). An N-group $_N\Gamma$ is monogenic if there is $\gamma \in \Gamma$ such that $N\gamma = \Gamma$. Three classes of monogenic N-groups are given–all of which coincide with irreducibility in the case of ring-modules. Type 0, $_N\Gamma$ is simple; Type 1, $_N\Gamma$ is simple and strongly monogenic (for all $\gamma \in \Gamma$, $N\gamma = \{0\}$ or Γ); Type 2, $_N\Gamma$ is N_0-simple (where Γ is considered as an N_0-group, N_0 zero symmetric, i.e., $n0 = 0$ for all $n \in N_0$). For an N-group $_N\Gamma$ and subsets Δ_1, Δ_2 of N, define the quotient $(\Delta_1 : \Delta_2)_N = \{n \in N | n\Delta_2 \subseteq \Delta_1\}$. A theorem of S. D. Scott is given: Suppose N is zero-symmetric, has the DCCN and M is a subgroup of N such that $NM \subseteq M$. If M is monogenic (by m_0) then M has a right identity and $(0, m_0)_M = \{0\}$. Another theorem of Scott is the following for zero-symmetric near-rings with DCCN: If I is a minimal ideal, then I is a finite direct sum of N-isomorphic minimal left ideals of N.

For a given near-ring one considers faithful and simple N-groups based upon N. For $\gamma = 0, 1, 2$, N is called γ-primitive on $_N\Gamma$ provided $_N\Gamma$ is faithful and of type γ. N is called γ-primitive provided there is Γ so that N is γ-primitive on $_N\Gamma$. An ideal I in N is γ-primitive provided N/I is γ-primitive. These notions of primitivity are crucial in a description of extensive (but still incomplete) work toward a density theorem for near-rings.

For a near-ring N and for $\gamma = 0, 1, 2$, consider N-groups Γ of type γ. Consider the intersection of sets $(n \in N | n\Gamma = 0)$ over such Γ. This intersection gives a γ-radical. Its "size" is indicative of how far N is from being "γ-semisimple." Characterizations of such radicals are given along with much other information concerning radicals in near-rings. The discussion of radicals completes the central part of the book–the part on structure theory.

Various special classes of near-rings are studied. Among these are distributively generated near-rings, transformation near-rings and near-fields. An element $d \in N$ is distributive if $d(n + n') = dn + dn'$ for all $n, n' \in N$. A near-ring is called distributively generated if there is a subsemigroup of $(N_d, \cdot)$ generating $(N, +)$ where N_d is the collection of distributive elements of N. Among other unsettled questions it is asked whether or not every zero-symmetric near-ring is embeddable into some distributively generated near-ring.

This book is an algebra book. Its subtitle is "The theory and its applications." The applications given are mainly to other parts of algebra and a few are given to geometry–not exactly what comes to mind to most these days when "applications" are mentioned. The reader may gather from the first part of this review that the reviewer feels there will eventually be a host of *real* applications–applications real in the sense that linear algebra has real applications.