end of the chapters which bring the reader up to date in the literature. Occasionally, there are omissions: the results on linearly ordered groups and rings have not received due attention in these remarks, e.g., real closed fields are totally ignored.

Rather modest background is needed to read the book; basic group and ring theory with some knowledge in lattice theory should suffice in general, but from time to time, additional knowledge is required (e.g., in the chapter on sheaf representations of lattice-ordered rings). The frequent motivations and the readable style make this volume a good choice for a graduate text.

The total impression about the material covered by this book is that, though the major motivation seemed to be more internal than external, there has been a commendable effort by the authors to relate their subject to other fields of mathematics. The theory of real functions lends its flavor throughout the subject, abstract group theory has penetrated so far only into the theory of totally ordered groups, while the few problems studied recently under the influence of modern ring and module theory have not had a great impact on the development of lattice-ordered structures (except for the attractive theory of free lattice-ordered groups). It is hoped that this excellent text will enhance the interest in lattice-ordered groups and rings, and their applications in various other fields.

LASZLO FUCHS

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General lattice theory, by George Grätzer, Mathematische Reihe Band 52, Birkhäuser Verlag, Basel, 1978, xiii + 381 pp.

Lattice theory has come a long way in the last 45 years! Although Dedekind had written two penetrating papers on "Dualgruppen" before 1900, and insightful isolated theorems had been published in the 1920's by individual mathematicians such as by R. Baer, K. Menger, F. Riesz, Th. Skolem, and A. Tarski, it was not until the 1930's that lattice theory became studied systematically, and recognized as a significant branch of mathematics.

This recognition was largely due to realization that "many mathematical theories may be formulated in terms of [lattice-theoretic concepts], and the systematic use of these concepts gives a unification and simplification of the various theories"[1] In brief, it was due to the wide range of *applications* of lattice theory to other branches of mathematics, and emphasis on such applications pervaded the talks given at the first symposium on lattice theory [1], sponsored by the American Mathematical Society in 1938.

Indeed, success may have come too easily to lattice theory in the first decade of its renaissance. The very simplicity and pervasiveness of its basic concepts (greatest lower and least upper bounds of order relations), and the ready availability of general ('universal') algebraic techniques having well-known analogues for groups and rings, made some mathematicians (most

[1]O. Ore in Bulletin of the American Mathematical Society **48** (1942), p. 75.

notably M. Bourbaki) suspect that the new subject lacked depth.

However, as with most other simple yet pervasive mathematical ideas (such as those of 'integral' and 'group') beneath the surface lay buried many deep questions. Is every finite lattice isomorphic with a sublattice of the lattice Π_n of all partitions of some *finite* set? Is every finite lattice isomorphic with the lattice of all congruence relations of some finite 'algebra'? Is a lattice in which every element has a unique complement necessarily distributive? How many elements does the free distributive lattice with n generators contain? How can one decide when two lattice polynomials in four symbols define the same function in every modular lattice? Is every complete, 'algebraic' (i.e., compactly generated) lattice isomorphic to the lattice of all congruence relations of some algebra?

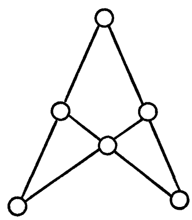
During the past 35 years, connoisseurs of lattice theory have increasingly concentrated their efforts on such internal problems of 'pure' lattice theory, and their efforts have been crowned with a series of brilliant successes; most of the above problems are now solved. Moreover the charm and simplicity of the ideas involved have stimulated the publication of at least 25 books, all advancing our understanding of the subject.

Of these, Professor Grätzer's new book on *General lattice theory* is a distinguished sequel to his *Universal algebra* (1968) and *Lattice theory*: *first concepts and distributive lattices*, (1971), which I shall refer to below as UA and FC. With its 900 exercises and bibliography of 750 entries, it constitutes an excellent introduction to 'pure' lattice theory for graduate students contemplating research in this field. More than that, it provides a comprehensive survey of much of the best research on this subject over the past 20 years. Written with precision and style, it is a distillation of eight years of seminars on lattice theory at the University of Manitoba, and of the author's experience as editor-in-chief of *Algebra Universalis*.

Its first two chapters, appropriately entitled 'first concepts' and 'distributive lattices', can be viewed as polishing and up-dating FC. Like FC, they culminate with an in-depth study of the structure of 'Stone lattices', such as arise in general topology and Brouwerian logic. The new version is supplemented by 8 pages on 'further topics' (essentially a guide to the research literature), and a list of 76 unsolved problems, many of them taken from FC.

The third chapter deals with the structure of lattices. It gives prominent roles to the concepts of 'standard' elements and ideals, and to 'weak' perspectivity and 'weak' modularity, concepts which the author and E. T. Schmidt invented nearly 20 years ago. It proves that every lattice that is relatively complemented *or* modular is 'weakly' modular, and that the concepts of 'distributive', 'neutral', and 'standard' elements are equivalent in any weakly modular lattice.

As a result, the distinction between them is unnecessary in modular and geometric lattices, to which the next chapter is largely devoted. This chapter is enlivened by an explanation of Whitney's graph-theoretic interpretation of 'matroids', and a detailed analysis of symmetric partition lattices and representations (of B. Jónsson's 'types' 1, 2, and 3) of general lattice by sublattices of partition lattices. An interesting innovation (due to H. Crapo) consists in associating with each geometric lattice the *edge geometry* of its points, but drawing in only these 'lines' which pass through three or more points. This enables one to reconstruct the lattice Π_4 of all partitions of the set $\mathbf{4} = \{1, 2, 3, 4\}$ from the simple configuration drawn above. Its six 'points' are the partitions which identify a single pair of elements; its four 'lines' correspond to the partitions of **4** into a triple and a singleton. The chapter also includes an incisive exposition of Jónsson's theory of 'arguesian' lattices, and of their coordinatization by division rings.



The last two chapters deal with the (distributive) 'lattice of equational classes of lattices', and 'free products' of lattices, respectively. These are active topics of contemporary research in lattice theory, and Professor Grätzer's well-knit exposition provides the best current survey of what is known about them. They conclude with a generalization of Dilworth's astonishing theorem (astonishing because of the long-standing conjecture that lattices with unique complements had to be distributive) that every lattice is a sublattice of a uniquely complemented lattice, and with the Grätzer-Sichler theorem that every monoid is isomorphic with some End$[L; \wedge, \vee, 0, I]$, the endomorphism monoid of some lattice with universal bounds.

The usefulness of *General lattice theory* for research workers is enhanced by detailed reviews of the current status (as of 1977!) of most of the research questions it discusses, and by lists of problems (193 in all!), at the end of each chapter. As a guess, these reviews and lists of problems should materially assist the writers of at least fifty Ph.D. Theses!!

Partly because of its emphasis on contemporary research, Professor Grätzer's latest book does not attempt to relate lattice theory to the fabric of mathematics as a whole. His tidy and self-contained exposition does not fully illuminate the subject with the deep insights which it has received from other branches of mathematics. Thus much of the inspiration for Dedekind's and the author's original work, and that for Ore's two classic papers of 1935–1936, came from intuitive pre-existing ideas (most notably the Jordan-Hölder Theorem) about groups and rings. Whitney's 'matroids' represented a synthesis of his ideas about cycles in graphs with his exposition of the elementary theory of linear dependence to Harvard undergraduates. Stone was concerned with the (bi)compactification problem in topology; Tarski with applications of Boolean algebra to logic; Menger with the classic idea that projective geometry dealt with properties of "Verbindung und Schnitten"; Weisner and Phillip Hall with the Möbius function of elementary number theory; and so on. *General lattice theory* does not bring out the historic *roots* of lattice theory in the soil of mathematics as a whole; it is only presented as a flowering subject; even its fruits are only briefly mentioned in passing, often as exercises.

To this reviewer, the time seems ripe for experts in lattice theory to reconsider the challenging question asked G. D. Birkhoff in 1933 [3, p. 7]. After listening attentively to my earnest explanation of some of their basic properties, he asked: "What can be proved using lattices that cannot be proved without them?" Not only was this question the main theme of the 1938 symposium [1] at which lattice theory was first given publicity by the American Mathematical Society, but its stimulus still strongly pervaded the much deeper 1960 symposium [2] on the same subject.

Instead, Professor Grätzer draws a careful line between "lattice theory proper and its allied fields", and avoids discussing results which "belong to universal algebra and not to lattice theory". This partly neutralizes his important but brief comment at the beginning of Chapter V, that "of the four characterizations given, three apply to arbitrary equational classes of universal algebras". Although it may be most efficient for the product of Ph.D. theses to subdivide mathematics up into neat, self-contained branches, the vitality of mathematics depends in the long run on a widespread familiarity with interconnections between these branches, and even on ideas coming from other areas of science.

Nevertheless, for those who already appreciate lattice theory, or who are curious about its techniques and intriguing internal problems, Professor Grätzer's lucid new book provides a most valuable guide to many recent developments. Even a cursory reading should provide those few who may still believe that lattice theory is superficial or naive, with convincing evidence of its technical depth and sophistication.

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Crystallographic groups of four-dimensional space, by Harold Brown, Rolf Bülow, Joachim Neubüser, Hans Wondratschek and Hans Zassenhaus, Wiley, New York, 1978, xiv + 443 pp., $38.90.

In Euclidean n-space, a *crystallographic space group* is a discrete group of isometries which contains, as a subgroup, the group generated by n independent translations. For this subgroup, which is abstractly C_∞^n (the free Abelian group with n generators), the orbit of any point is a *lattice*, which may alternatively be described as an infinite discrete set of points whose set of position vectors is closed under subtraction. The word 'crystallographic' is used because the positions of atoms in a crystal are well represented by lattices (with $n = 3$) or by sets of superposed lattices. For instance, the *cubic* lattice, of points whose Cartesian coordinates are integers, describes the