To this reviewer, the time seems ripe for experts in lattice theory to reconsider the challenging question asked G. D. Birkhoff in 1933 [3, p. 7]. After listening attentively to my earnest explanation of some of their basic properties, he asked: "What can be proved using lattices that cannot be proved without them?" Not only was this question the main theme of the 1938 symposium [1] at which lattice theory was first given publicity by the American Mathematical Society, but its stimulus still strongly pervaded the much deeper 1960 symposium [2] on the same subject.

Instead, Professor Grätzer draws a careful line between "lattice theory proper and its allied fields", and avoids discussing results which "belong to universal algebra and not to lattice theory". This partly neutralizes his important but brief comment at the beginning of Chapter V, that "of the four characterizations given, three apply to arbitrary equational classes of universal algebras". Although it may be most efficient for the product of Ph.D. theses to subdivide mathematics up into neat, self-contained branches, the vitality of mathematics depends in the long run on a widespread familiarity with interconnections between these branches, and even on ideas coming from other areas of science.

Nevertheless, for those who already appreciate lattice theory, or who are curious about its techniques and intriguing internal problems, Professor Grätzer's lucid new book provides a most valuable guide to many recent developments. Even a cursory reading should provide those few who may still believe that lattice theory is superficial or naive, with convincing evidence of its technical depth and sophistication.

REFERENCES

1. Symposium on Lattice Theory held April 15, 1938; published in Bull. Amer. Math. Soc. **44** (1938), 793–827.

2. *Lattice theory*, Proc. Sympos. Pure Math., vol. 2, R. P. Dilworth (ed.), Amer. Math. Soc., Providence, R. I., 1960, 208 pp.

3. *Trends in lattice theory*, J. C. Abbott (ed.), van Nostrand, New York, 1970.

GARRETT BIRKHOFF

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 5, September 1979

0002-9904/79/0000-0412/$01.75

Crystallographic groups of four-dimensional space, by Harold Brown, Rolf Bülow, Joachim Neubüser, Hans Wondratschek and Hans Zassenhaus, Wiley, New York, 1978, xiv + 443 pp., $38.90.

In Euclidean n-space, a *crystallographic space group* is a discrete group of isometries which contains, as a subgroup, the group generated by n independent translations. For this subgroup, which is abstractly C_∞^n (the free Abelian group with n generators), the orbit of any point is a *lattice*, which may alternatively be described as an infinite discrete set of points whose set of position vectors is closed under subtraction. The word 'crystallographic' is used because the positions of atoms in a crystal are well represented by lattices (with $n = 3$) or by sets of superposed lattices. For instance, the *cubic* lattice, of points whose Cartesian coordinates are integers, describes the structure of NaCl if we regard each lattice point (x, y, z) as the location of an atom of sodium or chlorine according to the parity of $x + y + z$. In the reviewer's notation [Coxeter **1940**, p. 402; 1978, p. 467] the cubic lattice is δ_4 and the *face-centred* cubic lattice (all the atoms of either kind alone) is $h\delta_4$. But the geometric notions are just as significant in any number of dimensions.

When $n = 1$ there is essentially only one lattice: its points are evenly spaced along a line. There are two 1-dimensional space groups: the infinite cyclic group C_∞ generated by one translation, and the infinite dihedral group D_∞ generated by two reflections.

The 2-dimensional space groups (which involve also rotations and glide-reflections) are the symmetry groups of wallpaper patterns and mozaics. Although these groups had long been used by artists, they were first precisely enumerated by E. S. Fedorov [**1891**] who found that there are just 17 of them. As they were rediscovered by F. Klein, G. Pólya, P. Niggli and others, it is not surprising to find that various notations have been published. These have been neatly collected and explained by Doris Schattschneider [**1978a**, p. 449]. It has often been alleged that all these groups were employed by the Moors in their decoration of the Alhambra, but in fact those artists used only eleven of the seventeen. Five of the remaining six (namely $p2$, pm, pg, pgg and $p3m1$) were used by the Bakuba and Benin tribes in Africa, south of the Sahara [Crowe **1971**; **1975**]. The last one, $p31m$, occurs in a Chinese pattern [Fejes Tóth **1964**, p. 40, Plate II.1].

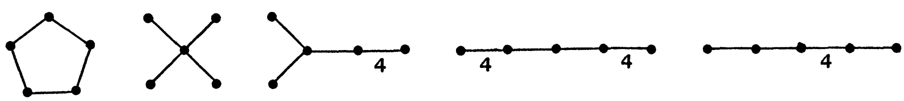
In such patterns, variation of color provides an extra feature which can likewise be treated mathematically. M. C. Escher went far beyond the Africans and Chinese by using animal shapes instead of geometric abstractions. Caroline MacGillavry [**1976**] was so much impressed by his intuitive discovery of many of the colored symmetry groups (without any mathematical erudition) that she persuaded him to illustrate, in his own special way, those groups which he had previously overlooked.

It is well known that congruent replicas of any triangle or quadrangle can be fitted together to fill and cover the plane. But the possible shapes of plane-filling pentagons have never been completely enumerated (not even the convex ones) [Schattschneider **1978b**]. The analogous enumeration of convex space-filling polyhedra has been initiated in a sequence of papers by Goldberg [**1978**]. It seems highly probable that the greatest possible number of faces for such a polyhedron is 16. The discovery of a space-filling 16-hedron was incorrectly attributed to Critchlow [Coxeter **1978**, p. 468]. Goldberg has pointed out this same solid was described much earlier by Föppl [**1914**].

Fedorov, who enumerated the 17 two-dimensional space groups, also enumerated the 219 three-dimensional space groups, with the help of Schoenflies (see Burckhardt [**1971**]). (The number is usually given as 230, but eleven of the 230 come in enantiomorphic pairs, differing only in the sense of the twists that occur.) The successful solution of this difficult problem by Fedorov, Schoenflies and Barlow, is often cited as a striking instance of independent discovery at about the same time by different researchers (like the discovery of non-Euclidean geometry). But in fact Fedorov and Schoenflies revised their tentative lists by exchange of letters, and Barlow's list, made several years later, was incomplete.

Wondering how the sequence 2, 17, 219 might continue, we need not be surprised to learn that there are no fewer than 4783 four-dimensional space groups. Their complete tabulation, in the book by Brown, Bülow, Neubüser, Wondratschek and Zassenhaus, must be hailed as an outstanding achievement. In connection with the 227 holohedries, they refer to É. Goursat's work of 1889 on the finite groups of isometries in elliptic 3-space, but they do not mention the extension to spherical 3-space by Threlfall and Seifert [**1931**] and its clarification by P. DuVal [**1964**, pp. 57–61]; see also Coxeter [**1974**, Chapter 6]. It is possible that they might have been helped by such an application of quaternions.

Some of the most interesting crystallographic groups, in any number of dimensions, are subgroups of the discrete groups generated by *reflections* [Coxeter **1934**, p. 619; Bourbaki **1968**, p. 199; Flatto **1978**, p. 260] which were enumerated in Princeton while Hermann Weyl was developing Élie Cartan's ideas on semisimple Lie groups. The graphical symbols used in that enumeration were rediscovered (in a slightly different form) by E. B. Dynkin (see Coxeter [**1978**, p. 466]). The 4-dimensional groups



(each generated by five reflections, represented by the five vertices of the graph) must occur somewhere among the 4783, but it is a considerable task to locate them.

BIBLIOGRAPHY

N. Bourbaki [**1968**], *Groupes et algèbres de Lie, Chapitres* 4, 5 *et* 6, Hermann, Paris.

J. J. Burckhardt [**1971**], *Der Briefwechsel von E. S. von Fedorow und A. Schoenflies*, 1889–1908, Archive for History of Exact Sciences 7, 91–141.

H. S. M. Coxeter [**1934**], *Discrete groups generated by reflections*, Ann. of Math. **35**, 588–621.

______ [**1940**], *Regular and semi-regular polytopes*, Math. Z. **46**, 380–407.

______ [**1974**], *Regular complex polytopes*, Cambridge Univ. Press, London and New York.

______ [**1978**], *Review of Three-dimensional nets and polyhedra, by A. F. Wells*, Bull. Amer. Math. Soc. **84**, 466–470.

D. W. Crowe [**1971**], *The geometry of African art*: I, J. Geometry **1**, 169–182.

______ [**1975**], *The geometry of African art*: II, Historia Math. **2**, 253–271.

Patrick DuVal [**1964**], *Homographies, quaternions and rotations*, Clarendon Press, Oxford.

E. S. Fedorov [**1891**], Zap. Imp. S. Petersburg Mineral. OBŠČ. (2) **28**, 345–390.

L. Fejes Tóth [**1964**], *Regular figures*, Pergamon Press, New York.

Leopold Flatto [**1978**], *Invariants of finite reflection groups*, L'Enseignement Math. (2) **24**, 237–292.

Ludwig Föppl [**1914**], *Der Fundamentalbereich des Diamantgitters*, Physikalische Z. **15**, 191–193.

Michael Goldberg [**1978**], *On the space-filling heptahedra*, Geometriae Dedicata **7**, 175–184.

Caroline H. MacGillavry [**1976**], *Fantasy and Symmetry–The periodic drawings of M. C. Escher*, Abrams, New York.

Doris Schattschneider [**1978a**], *The plane symmetry groups: their recognition and notation*, Amer. Math. Monthly **85**, 439–450.

______ [**1978b**], *Tiling the plane with congruent pentagons*, Math. Magazine **51**, 29–44.

W. Threlfall and H. Seifert [**1931**], *Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen der driedimensionalen sphärischen Raumes*, Math. Ann. **104**, 1–70.

H. S. M. COXETER