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Introduction to mathematical economics, by Murray C. Kemp and Yoshio Kimura, Springer-Verlag, New York-Heidelberg-Berlin, 1978, vi + 249 pp., $19.80.

Among the Social Sciences, Economics is easily the most mathematically developed. This development has been going on for a hundred years, beginning with Walras in 1874 **[41]** and Edgeworth in 1881 **[10]**, and has been greatly accelerated during the past thirty years, spurred on by parallel developments in such areas of mathematics as convexity, game theory, linear programming, fixed point theory, measure theory and, more recently, nonstandard analysis and global analysis.

To attempt to describe the whole of Mathematical Economics is beyond the scope of this review, and of this reviewer. I will try, instead, to give some idea how these various mathematical disciplines come to play in the important branch of economic theory known as general equilibrium theory. Though the model discussed will be a greatly simplified one, it carries much of the flavor of the general theory.

We consider an economy in which n economic agents are involved in a trading situation involving m distinct goods. The basic data of the economy consist of nonnegative vectors $w^i \in \mathbf{R}^m_+$, telling how much of each good is owned *initially* by agent i, $i = 1, \ldots, n$, and *utility functions* u_i: $\mathbf{R}^m_+ \to \mathbf{R}$, describing the preferences of each agent over various combinations of the goods (vector x is preferred to y by agent i if $u_i(x) > u_i(y)$). The market process involves exchange of goods among the agents with each agent attempting to obtain a vector of goods which is more preferred. The end result of such trading is a collection of nonnegative vectors $x^i \in \mathbf{R}^m_+$, representing each agent's *final* holdings, which satisfies $\sum_{i=1}^n x^i \leqslant \sum_{i=1}^n w^i$. This condition expresses the fact that there is no possibility of production and so what leaves the market must have been brought in by some agent.

Now suppose prices are imposed on the market. That is, there is a vector $p \in \mathbf{R}^m_+$, $p \neq 0$, by which the agents can evaluate the relative worth of the goods. Thus, each agent can afford to buy any vector of goods in his budget set $B_i(p) = \{x \in \mathbf{R}^m_+ | \langle x, p\rangle \leqslant \langle w^i, p\rangle\}$. Motivated by his own preferences, agent i will seek a vector x^i which maximizes $u_i(x)$ over $B_i(p)$. We say the price vector p is a *eqilibrium price vector* if the resulting maximizing vectors $x^1, \ldots, x^n$ satisfy the feasibility condition $\sum x^i \leqslant \sum w^i$. That is, by imposition of the prices p, the agents, acting individually and selfishly, will be lead to an outcome which is feasible for the economy as a whole. In this case, we call the set of vectors $\{x^1, \ldots, x^n\}$ a *competitive allocation*.

A fundamental question is that of the existence of an equilibrium price vector. Walras, in his original formulation of the problem, inconclusively asserted the existence of an equilibrium on the basis of an equation counting argument. It was not until the 1930s that a correct proof of existence for a special case was given by Wald **[39]**, **[40]**. The general question received a satisfactory solution only in the 1950s, first by Arrow and Debreu in 1954 **[1]**, and independently by Gale [11], Kuhn [15], McKenzie [18], [19], and Nikaido [22]. In our special case, the Arrow-Debreu result has the following form. Suppose each u_i is concave, continuous and does not attain a maximum value on $\mathbf{R}^m_+$, and suppose, further, that each $w^i > 0$. Then there exists an equilibrium price vector p.

The original proof of Arrow and Debreu (whose model included production as well as trading) consisted of reformulating the problem to be that of finding an equilibrium point in a certain abstract game, and deducing existence from a generalization of a game theoretic result of Nash [21]. The Nash result relied on the Kakutani fixed point theorem [14], a generalization of the Brouwer fixed point theorem to convex-valued point-to-set maps. It was soon seen that the economic equilibrium question could be deduced directly from the Kakutani theorem by the construction of a suitable point-to-set map. These ideas form the basis for recent methods of Scarf [25] and others for computing economic equilibria by finding fixed points of maps (see also [35]). It is interesting to note here that the first uses of the Brouwer theorem in Economics were probably made by von Neumann in his proof of the minimax theorem in 1928 [36] and in his treatment of a model of an expanding economy in 1937 [37].

Returning to our simple trading economy, suppose $\{x^1, \ldots, x^n\}$ is a set of feasible final holdings (a reallocation of $\{w^1, \ldots, w^n\}$) such that no subgroup of the traders could reallocate their *own* initial holdings in such a way as to give them each a more preferred outcome. That is, for each $S \subseteq \{1, \ldots, n\}$, there is *no* set $\{y^i \in \mathbf{R}^m_+ | i \in S\}$ such that $\Sigma_{i \in S} y^i \leqslant \Sigma_{i \in S} w^i$ and $u_i(y^i) > u_i(x^i)$ for $i \in S$. Such an allocation $\{x^1, \ldots, x^n\}$ is said to be in the *core* of the economy, and has the property that no subset of the traders will benefit from leaving the economy and trading among themselves.

This notion of a stable reallocation comes directly from the theory of n-person games, and in 1959 Shubik [27] pointed out the relationship of the core to a conjecture of Edgeworth almost 80 years before. It is easy to show that any competitive allocation is in the core. Edgeworth essentially conjectured (for the case $n = 2$) that if you form an economy with nk traders by replacing each original trader by k copies of himself, then as $k \to \infty$, the core "shrinks" to the set of competitive allocations. In 1963, Debreu and Scarf [9] proved this result. What this says is that as the economy in some sense grows large, if traders merely seek the sort of stability implied by the core, they will in any case be led to allocations having equilibrium prices.

The next step in the process was initiated by Aumann in 1964 [2], when he proposed the notion of a market with a *continuum* of traders as the way to model the notion of perfect competition (i.e., individual readers cannot influence the prices) which is implicit in the general equilibrium analysis. Each trader is a point in a nonatomic measure space, and, having measure zero, is without individual influence. This infinite model of a large economy is like the continuous model of a fluid with a large finite number of molecules. (Games with a continuum of players had already been considered by Milnor and Shapley [20] and Shapley [26].) Aumann proved that for such an economy, the set of core allocations is equal to the set of equilibrium allocations, thus giving an "in the limit" version of the Debreu-Scarf limit theorem. Later [3] he showed that under mild assumptions, these markets always had competitive prices, and so these two sets are in fact not empty. It is interesting to note that he was able to dispense with the assumption of convexity of each agent's preferences (provided in our example by concave utility functions); the required convexity properties are provided instead by Lyapunov's theorem, which states that the range of a nonatomic, vector-valued measure is always convex (and compact) [17]. Measure theoretic models of games and economies have received considerable attention in recent years; see, for example, the recent books by Aumann and Shapley [4] and Hildenbrand [13]. In a related vein, Brown [5] and others have treated limit theorems for large economies using the methods of nonstandard analysis.

The theoretical breakthroughs of the fifties depended in a large part on the use of the methods of convexity, methods essentially introduced to the subject in 1944 by von Neumann and Morgenstern in their work on game theory [38]. (The significance of convexity in nonlinear optimization was first noted by Kuhn and Tucker [16].) It was generally felt (see, for example, the introduction to the book *Convex structures and economic theory* by Nikaido [23]) that the mathematics of convexity, including the related areas of linear programming and game theory, was to Economics as the calculus is to the physical sciences. In his 1959 book, Debreu cites the work of von Neumann and Morgernstern "which freed mathematical economics from its traditions of differential calculus and compromises with logic." He noted the change in methods to be "essentially a change from the calculus to convexity and topological properties, a transformation which has resulted in notable gains in the generality and in the simplicity of the theory" [6, p. x].

However, over the past 10 years, this point of view seems to have reversed itself. In 1970, Debreu [7] showed that if the preferences of the market can be specified in a suitably differentiable manner, then for almost all choices of the initial holdings $w^1, \ldots, w^n$, the market will have only a finite number of equilibrium price vectors. The main result used in his proof was Sard's theorem [24] on the measure of the set of critical values of a differentiable map. Since then there has been much work in this vein, including many contributions by Smale and his students. (See the sequence of papers "*Global analysis and economics*. I–VI" by Smale, [28]–[33]. All but one of these have appeared in the relatively new *Journal of Mathematical Economics*, a good place to look for any mathematician interested in learning the current state of the mathematics of economics.)

Unfortunately, the book of Kemp and Kimura does not deal with any of the matters that have been discussed so far. It is mainly a book about nonnegative matrices and stability of systems of ordinary differential equations. While there is a first chapter on linear inequalities, covering such topics as linear and nonlinear programming, matrix games, and some elementary facts about polyhedral convex sets, this material does not come to play later in this book and seems more to have been added as an afterthought. There are two chapters on nonnegative matrices, a chapter on stability analysis of systems of diferential and difference equations, and two chapters (in total pages, less than one fifth of the book) in which certain economic models are discussed. It is troubling that with all the emphasis on nonnegative matrices, the von Neumann growth model is not included.

Aside from questions of taste concerning the choice of topics, I find the title of this book to be misleading. It is not really an introduction to the economic models that are discussed; one needs to have some prior idea of what these models are about in order to understand the treatment given by the authors. There is almost no discussion of the economic meaning of their definitions and results. Nor is it a good introduction to the mathematical topics discussed. Many of their proofs of elementary results are longer and more cumbersome than necessary. In at least one place, they say something that is just wrong: Lemma 1 on page 116 states that a certain set of functions is a subspace *of* $\mathbf{R}^n$.

For a good introductory treatment of a broad range of topics in mathematical economics, one can consult the book of Nikaido [23], or the more recent book by Takayama [34]. For a good elementary treatment of the geometry of linear inequalities and linear programming (which treats some simple economic models, including the von Neumann model), the 1960 book of Gale [12] is still my favorite. For a technical survey of equilibrium theory with many references, see Debreu's address to the 1974 Congress in Vancouver [8]. Finally, one should be on the lookout for the forthcoming *Handbook of mathematical economics*, edited by Arrow and Intrigilgator.

But what of the book of Kemp and Kimura? For someone already familiar with the basic mathematics and economics involved, it can serve as an exhaustive and up-to-date reference work. For many of the results presented, there is an extensive discussion of generalizations and variations, together with many recent references to the literature. In this sense, for certain specialists, the book can prove to be quite useful.

References

1. K. J. Arrow and G. Debreu, *Existence of an equilibrium for a competitive economy*, Econometrica **22** (1954), 265–290.

2. R. J. Aumann, *Markets with a continuum of traders*, Econometrica **32** (1964), 39–50.

3. ______, *Existence of competitive equilibria in markets with a continuum of traders*, Econometrica **34** (1966), 1–17.

4. R. J. Aumann and L. S. Shapley, *Values of non-atomic games*, Princeton Univ. Press, Princeton, N. J., 1974.

5. D. J. Brown, *Existence of a competitive eqilibrium in a non-standard exchange economy*, Econometrica **44** (1976), 537–546.

6. G. Debreu, *Theory of value*, John Wiley, New York, 1959.

7. ______, *Economies with a finite set of equilibria*, Econometrica **38** (1970), 387–392.

8. ______, *Four aspects of the mathematical theory of economic equilibrium*, Proc. Internat. Congr. Mathematicians, Vancouver, 1974.

9. G. Debreu and H. Scarf, *A limit theorem on the core of an economy*, Internat. Econom. Rev. **4** (1963), 235–246.

10. F. Y. Edgeworth, *Mathematical psychics*, Kegan Paul, London, 1881.

11. D. Gale, *The law of supply and demand*, Math. Scand. **3** (1955), 155–169.

12. ______, *The theory of linear economic models*, McGraw-Hill, New York, 1960.

13. W. Hildenbrand, *Core and equilibria of a large economy*, Princeton Univ. Press, Princeton, N. J., 1974.

14. S. Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Math. J. **8** (1941), 457–458.

15. H. W. Kuhn, *On a theorem of Wald*, Linear Inequalities and Related Systems, H. W. Kuhn and A. W. Tucker, eds., Princeton Univ. Press, Princeton, N. J., 1956, 265–273.

16. H. W. Kuhn and A. W. Tucker, *Non-linear programming*, Proc. Second Berkeley Sympos. on Mathematical Statistics and Probability, J. Neyman, ed., University on California Press, Berkeley, Calif. 1951.

17. A. Lyapunov, *Sur les fonctions-vecteurs complètement additives*, Bull. Acad. Sci. USSR, Sèr. Math. **4** (1940), 465–478.

18. L. W. McKenzie, *On equilibrium in Graham's model of world trade and other competitive systems*, Econometrica **22** (1954), 147–161.

19. ______, *On the existence of general equilibrium for a competitive market*, Econometrica **27** (1959), 54–71.

20. J. W. Milnor and L. S. Shapley, *Values of large games*. II: *Oceanic Games*, RM-2649, The Rand Corporation, Santa Monica, Calif., February 1961 (reprinted in Math. Oper. Res. **3** (1978), 290–307).

21. J. F. Nash, *Equilibrium points in n-person games*, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 48–49.

22. H. Nikaido, *On the classical multilateral exchange problem*, Metroecon. **8** (1956), 135–145.

23. ______, *Convex structures and economic theory*, Academic Press, New York, 1968.

24. A. Sard, *The measure of the critical points of differentiable maps*, Bull. Amer. Math. Soc. **48** (1942), 883–890.

25. H. Scarf (with T. Hansen), *The computation of economic equilibria*, Yale Univ. Press, New Haven, Conn., 1973.

26. L. S. Shapley, *Values of large games*. III: *A corporation with two large stockholders*, RM-2650, The Rand Corporation, Santa Monica, Calif., December 1961.

27. M. Shubik, *Edgeworth market games*, Contributions to the Theory of Games. IV, A. W. Tucker and R. D. Luce, eds., Princeton Univ. Press, Princeton, N. J., 1959.

28. S. Smale, *Global analysis and economics*. I, *Pareto optimum and a generalization of Morse theory*, Salvador Symposium on Dynamical Systems, Academic Press, New York, 1973.

29. ______, *Global analysis and economics*. IIA, *Extension of a theorem of Debreu*, J. Math. Econom. **1** (1974), 1–14.

30. ______, *Global analysis and economics*, III, *Pareto optima and price equilibria*, J. Math. Econom. **1** (1974), 107–117.

31. ______, *Global analysis and economics*. IV, *Finiteness and stability of eqilibria with general consumption sets and production*, J. Math. Econom. **1** (1974), 119–127.

32. ______, *Global analysis and economics*. V, *Pareto theory with constraints*, J. Math. Econom. **1** (1974), 213–221.

33. ______, *Global analysis and economics*. VI, *Geometric analysis of Pareto optima and price equilibria under classical hypotheses*, J. Math. Econom. **3** (1976), 1–14.

34. A. Takayama, *Mathematical economics*, Dryden Press, Hinsdale, Illinois, 1974.

35. M. J. Todd, *The computation of fixed points and applications*, Lecture Notes in Economics and Mathematical Systems, vol. 124, Springer-Verlag, Berlin, Heidelberg, New York, 1976.

36. J. von Neumann, *Zur Theorie der Gesellschaftsspiele*, Math. Ann. **100** (1928), 295–320.

37. ______, *Über ein Ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes*, Ergebnisse eines Mathematischen Kolloquiums **8** (1937), 73–83. [Translated as *A model of general economic equilibrium*, Rev. Economic Studies **13** (1945), 1–9.]

38. J. von Neumann and O. Morgenstern, *Theory of games and economic behavior*, Princeton Univ. Press, Princeton, N. J., 1944.

39. A. Wald. *Über die eindeutige positive Lösbarkeit der neuen Produktions gleichungen*, Ergebnisse eines Mathematischen Kolloquiums **6** (1935), 12–20.

40. ______, *Über einige Gleichungssysteme der Mathematischen Ökonomie*, Z. Nationalokonomie, no. 7. [Translated as *On some systems of equations of mathematical economics*, Econometrica **19** (1951), 368–403.]

41. L. Walras, *Eléments d'économie politique pure*, Corbaz, Lausanne. [Translated as *Elements of pure economics*, Trans. W. Jaffe, Allen and Unwin, London, 1954.]

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