provides a concise summary of material, called "functional support," to which the author refers in the remaining chapters on Padé approximants. Very few proofs are given for known classical results. Although it is questionable whether his new formulation of the concept of Padé approximants will be uniformly adopted, it seems certain that Gilewicz' treatment of the subject will be widely used and cited.

REFERENCES

1. George A. Baker, Jr., *Essentials of Padé approximants*, Academic Press, New York, 1975.

2. G. A. Baker, Jr. and John L. Gammel, Eds., *The Padé approximant in theoretical physics*, Academic Press, New York, 1970.

3. C. Brezinski, *A bibliography on Padé approximation and related subjects*, Publ. No. 96 du Lab. de Calcul, Univ. de Lille (1977).

4. H. Cabannes, *Padé approximant method and its applications in mechanics*, Lecture Notes in Physics, vol. 47, Springer-Verlag, New York, 1976.

5. P. R. Graves-Morris, (Ed.), *Padé approximants and their applications*, Proceedings of a Conference held at the University of Kent, Canterbury, England, (July, 1972), Academic Press, New York, 1973.

6. P. Henrici, *Applied and computational complex analysis*, vol. 2, Special Functions, Integral Transforms, Asymptotics and Continued Fractions, John Wiley and Sons, New York, 1977.

7. William B. Jones and W. J. Thron (Eds.), Proceedings of the International Conference on Padé Approximants, Continued Fractions and Related Topics, Rocky Mountain J. Math. **4** (2) (Spring 1974c), 135–397.

8. ______, *Continued fractions*: *Analytic theory and applications*, Addison-Wesley Pubishing Company, Inc. (to appear).

9. E. B. Saff and R. W. Varga, (Eds.), *Padé and rational approximation theory and application*, Academic Press, Inc., New York, 1977.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 2, March 1980

0002-9904/80/0000-0115/$02.00

Barrelledness in topological and ordered vector spaces, by T. Husain and S. M. Khaleelulla, Lecture Notes in Math., vol. 692, Springer-Verlag, Berlin-Heidelberg-New York, 1978, x + 258 pp.

This book collects a vast number of facts that are scattered over the literature. Its subject can be divided into two more or less independent parts. The first part, which takes up about two thirds of the book, is concerned with topological vector spaces exclusively, and the second part with ordered topological vector spaces. This review will be divided into two parts accordingly.

PART I. In the proof of the classical Banach-Steinhaus theorem and the closed graph theorem, the category argument is used to establish the following:

(*) If U is an absorbing, convex, circled subset of a Banach space E, then the closure $\overline{U}$ of U is a neighborhood of 0 in E. (For subsets A and B of a linear space, A is said to *absorb* B if there exists a positive number λ_0 such that $B \subset \lambda A$ for all $\lambda \geqslant \lambda_0$. The set A is called *absorbing* if A absorbs each point of the linear space.) Recognizing that the class of locally convex (l.c.) spaces E for which statement (*) is valid is much larger than the class of Banach spaces, Bourbaki [1] turned (*) into a definition: A closed, absorbing, convex, circled subset U of an l.c. space E is called a *barrel*, and the space E is called *barrelled* if each barrel in E is a neighborhood of 0.

True to expectation, barrelled spaces became the favorite domain spaces for various generalizations of Banach's closed graph theorem. The class of barrelled spaces is, in fact, the collection of all l.c. spaces suitable as the domain spaces in closed graph theorems. For, according to Mahowald **[10]**, an l.c. space E is barrelled if and only if each linear transformation from E into an arbitrary Banach space is continuous provided the graph of the transformation is closed. Are there equally natural classes of range spaces? Let $\mathfrak{S}$ be a family of l.c. spaces, and let $\mathfrak{R}(\mathfrak{S})$ be the collection of all l.c. spaces F such that the closed graph theorem is valid for linear transformations from an arbitrary member of $\mathfrak{S}$ into the space F (i.e. a linear transformation T: $E \to F$ is continuous whenever $E \in \mathfrak{S}$ and the graph of T is closed). There has been much research done on $\mathfrak{R}(\mathfrak{C})$, where $\mathfrak{C}$ is the class of all barrelled spaces (e.g. A. P. Robertson and W. J. Robertson **[12]**, and Pták **[11]**), and a definitive characterization of members of $\mathfrak{R}(\mathfrak{C})$ was given by Kōmura **[8]**. However, more recent research shows that $\mathfrak{R}(\mathfrak{C})$ can be substantially enlarged to include important l.c. spaces in applications by replacing $\mathfrak{C}$ with the smaller class $\mathfrak{B}$ of Banach spaces. Nor does this result in too drastic a loss of generality. For, if $\mathfrak{U}$ denotes the class of all inductive limits of Banach spaces, then $\mathfrak{R}(\mathfrak{B}) = \mathfrak{R}(\mathfrak{U})$, and the class $\mathfrak{U}$ is quite rich. (De Wilde, whose contribution to these developments is decisive, has written a very readable account of the subject **[2]**.)

The study of barrelled spaces, apart from its utility in formulating closed graph theorems, has also produced interesting results in itself in the past three decades. One of the most interesting of these results was proved comparatively recently by Valdivia **[15]**, and Saxon and Levin **[14]** independently. If a subspace (not necessarily closed) of a barrelled space is of countable codimension, then it is again barrelled. (For subspaces of finite codimension this inheritance property was proved by Dieudonné **[4]**.) But, in addition, the study of barrelled spaces has inspired studies of more general spaces–even of non-locally-convex analogs–and it is these more general spaces that form the subject of the first part of the book under review.

In order to see how one can generalize barrelled spaces, let us recall that the notion of barrelled spaces was introduced by abstracting a part of the proof of the Banach-Steinhaus theorem. This theorem states that a subset of the dual of a Banach space is equicontinuous if it is weak* (i.e. pointwise) bounded. The conclusion of this theorem, in fact, characterizes barrelled spaces. An l.c. space E is barrelled if and only if each weak* bounded subset of the dual E' is equicontinuous. Thus the class of barrelled spaces E is characterized by the fact that the Banach-Steinhaus theorem is valid for E. By requiring properties weaker than the Banach-Steinhaus theorem, it is thus possible to define classes of l.c. spaces larger than the class of barrelled spaces.

Let us give some examples.

(1) The class of spaces E such that the evaluation map of E into its second dual is continuous (a fact we take for granted for Banach spaces) is characterized by the property that each strongly bounded subset of E' is equicontinuous (a subset of E' is *strongly* bounded if it is uniformly bounded on each bounded subset of E; clearly a strongly bounded set is weak* bounded). The space E with this property is called *quasibarrelled.*

(2) The $(\mathfrak{D}\mathfrak{F})$-spaces of Grothendieck [5] abstracting properties of the duals of metrizable l.c. spaces. An l.c. space is a $(\mathfrak{D}\mathfrak{F})$*-space* if

(a) there is a sequence $\{B_n: n \in \mathbf{N}\}$ of bounded subsets of E such that each bounded subset is contained in some B_n and

(b) a countable union of equicontinuous subsets of E' is again equicontinuous provided it is strongly bounded.

Condition (b) above is a much weakened form of the Banach-Steinhaus theorem.

(3) Husain [6] defines two additional classes of l.c. spaces generalizing barrelled (resp. quasibarrelled) spaces: an l.c. space E is *countably barrelled* (resp. *countably quasibarrelled*) if a countable union of equicontinuous subsets of E' is again equicontinuous provided it is weak* (resp. strongly) bounded. Note that condition (b) in the definition of $(\mathfrak{D}\mathfrak{F})$-spaces is equivalent to E being countably quasibarrelled. In particular, the dual of a metrizable l.c. space is always countably quasibarrelled with respect to the strong topology, (i.e. the topology of uniform convergence on bounded subsets of E).

So far we have only considered l.c. spaces. Are there non-l.c. analogs of, say, barrelled or quasibarrelled spaces? Definitions in terms of equicontinuous sets are inappropriate, for there are topological vector spaces with trivial dual. For l.c. spaces E, however, the conditions on equicontinuous subsets of E' (i.e. Banach-Steinhaus-like properties) that we have mentioned can all be translated into conditions for barrels in E to be neighborhoods of 0. (Examples: (1) The original definition of barrelled spaces was in terms of barrels; (2) the characterizing property of quasibarrelled spaces E is equivalent to: a barrel in E is a neighborhood of 0 if it absorbs each bounded subset of E.) Though these conditions on barrels make perfect sense in arbitrary (not necessarily l.c.) topological vector spaces, for a non-l.c. space the conditions may have little to do with the topology, since barrels are convex by definition. To remedy this defect, Iyahen [7] introduced the notion of ultrabarrels (a notion anticipated in Robertson [13]). A closed, circled subset B_0 of a topological vector space $(E, \mathfrak{U})$ is called an *ultrabarrel* if there exists a sequence $\{B_n: n \geqslant 1\}$ of closed, circled, absorbing subsets of E such that $B_n + B_n \subset B_{n-1}$ for all $n \geqslant 1$ or, equivalently, if there exists a vector topology $\mathcal{V}$ for E (i.e. a topology $\mathcal{V}$ with which E is a topological vector space) such that B_0 is a $\mathcal{V}$-neighborhood of 0 and $\mathfrak{U}$-closed $\mathcal{V}$-neighborhoods of 0 form a local base for $\mathcal{V}$. Note that a barrel B is an ultrabarrel (just consider $\{2^{-n}B: n \geqslant 1\}$), but not conversely–the prefix "ultra-" is misleading. Using ultrabarrels, it is possible to define analogs of barelled, quasibarrelled, countably barrelled and countably quasibarrelled spaces. For example, a topological vector space is *ultrabarrelled* if each ultrabarrel is a neighborhood of 0. (Note that an l.c. ultrabarrelled space is barrelled, but an l.c. barrelled space need not be ultrabarrelled.) The theory of these "ultra-" spaces mimics that of the corresonding l.c. spaces, but it is not as rich.

Let us now turn to the book under review.

Chapter I (41 pages) is a review of basic notions and results from the theory of topological vector spaces and ordered topological vector spaces. All proofs are omitted in this chapter. Chapters II, V, and VI (103 pages total) are devoted to a study of various generalizations of barrelled spaces, including those discussed above. Chapters III and VII (35 pages total) treat those "ultra-" spaces mentioned above. The presentation of the material resembles that of an encyclopedia. There are headings (names of classes of topoogical vector spaces), and each heading is followed by a compendium of facts concerning the spaces in question. The writing is lucid, and the proofs are easy to follow, and, in most of the cases, self-contained.

Our criticism of this part of the book concerns its organization. In a survey such as this, we might hope for a unifying point of view over the mass of results scattered in the literature published in the past thirty years, and we might also hope that the authors would take advantage of the progress made during the period by providing the readers with simpler proofs and more general theorems. This book does not meet our hopes. The rigid formal style militates against any real unity of presentation, and in too many cases the material is taken directly from the original papers without regard to better ways that are available elsewhere.

Here are two examples of what we mean.

EXAMPLE 1. The theorem of Valdivia and Saxon-Levin, which states that the property of being barrelled is transmitted to subspaces of counable codimension, appears as Theorem 9 in Chapter II with Valdivia's proof **[15]**. The proof of the analogous fact for countably barrelled spaes, due to Webb **[17]**, appears in Chapter V. This proof uses Lemma 1 of the *next* chapter (Chapter VI). The analogous theorem for σ-barrelled spaces, due to Levin and Saxon **[9]**, is proved in Chapter VI. The proof uses Lemma 1 again.

Now this Lemma 1 is also the major step in the Saxon-Levin proof of Theorem 9, Chapter II (a proof quite different from Valdivia's proof given in Chapter II). It would have been clearer and more economical to treat the subject of subspaces of countable codimension in one place instead of spreading it out with much redundancy over three separate chapters.

Also the key lemma–Lemma 1, Chapter VI–is given the original proof of **[14]**. And the authors add that a shorter proof is available in **[17]**. Why did they reject the delightful proof in **[17]** in favor of a much longer one?

EXAMPLE 2. Grothendieck's theorem on the localization of the topology in $(\mathfrak{D}\mathfrak{F})$-spaces [**5**, Theorem 3] is given in Chapter II with the original proof. Yet in Chapter V a more general theorem with a subtler proof due to DeWilde and Houet [3] is presented. This theorem of DeWilde and Houet on countably barrelled spaces, besides being of interest in itself, is a common generalization of Grothendieck's theorem on $(\mathfrak{D}\mathfrak{F})$-spaces [ibid] and Valdivia's theorem on barrelled spaces [**15**, Theorem 5]. This point seems to be ignored. Why?

PART II. Chapters IV, VIII, and IX (65 pages total) of the book treat ordered topological vector lattices for the most part, while referring occasionally to more general ordered topological vector spaces.

Let $(E, C, \mathfrak{U})$ be an ordered topological space (i.e. $(E, \mathfrak{U})$ is a real topological vector space, and E has a partial ordering $\geqslant$ compatible with the linear structure of E, with $C = \{x: x \geqslant 0\}$). A subset B of E is called *order-bounded* if there exist x and y in E such that $x \leqslant z \leqslant y$ for all z in B.

If in the earlier discussion of topological vector spaces one replaces "bounded" with "order-bounded", one obtains various new classes of ordered topological vector spaces. For example, an ordered l.c. vector lattice E is called *order-quasibarrelled* if each barrel in E is a neighborhood of 0 provided it absorbs each order-bounded subset of E (this class was introduced by Wong **[18]**; see also **[19]**). The authors go on to define more and more complicated classes of ordered spaces with correspondingly longer names. (Our favorite is "*countably order-quasiultrabarrelled vector lattices*", which appears–all too briefly–in the heading of Chapter VIII.) As in the first part, the presentation of the material is formal, with emphasis on a multiplicity of definitions and elementary properties.

REFERENCES

1. N. Bourbaki, *Sur certains espaces vectoriels topologiques*, Ann. Inst. Fourier (Grenoble) **2** (1950), 5–16.

2. M. DeWilde, *Closed graph theorems and webbed spaces*, Research Notes in Math., vol. 19, Pitman, London, San Francisco, Melbourne, 1978.

3. M. DeWilde and C. Houet, *On increasing sequences of absolutely convex sets in locally convex spaces*, Math. Ann. **192** (1971), 257–261.

4. J. Dieudonné, *Sur les propriétés de permanence de certains espaces vectoriels topologiques*, Ann. Soc. Polon. Math. **25** (1952), 50–55.

5. A. Grothendieck, *Sur les espaces* ($\mathfrak{F}$) *et* ($\mathfrak{D}\mathfrak{F}$), Summa Brasil. Math. **3** (1954), 57–123.

6. T. Husain, *Two new classes of locally convex spaces*, Math. Ann. **166** (1966), 289–299.

7. S. O. Iyahen, *On certain classes of linear topological spaces*, Proc. London Math. Soc. (3) **18** (1968), 285–307.

8. Y. Kōmura, *On linear topological spaces*, Kumamoto J. Sci. Ser. A **5** (1962), 148–157.

9. M. Levin and S. Saxon, *A note on the inheritance of properties of locally convex spaces by subspaces of countable codimension*, Proc. Amer. Math. Soc. **29** (1971), 97–102.

10. M. Mahowald, *Barrelled spaces and the closed graph theorem*, J. London Math. Soc. **36** (1961), 108–110.

11. V. Pták, *Completeness and the open mapping theorem*, Bull. Soc. Math. France **86** (1958), 41–74.

12. A. P. Robertson and W. J. Robertson, *On the closed graph theorem*, Proc. Glasgow Math. Assoc. **3** (1956), 9–12.

13. W. J. Robertson, *Completions of topological vector spaces*, Proc. London Math. Soc. **8** (1958), 242–257.

14. S. Saxon and M. Levin, *Every countable-codimensional subspace of a barrelled space is barrelled*, Proc. Amer. Math. Soc. **29** (1971), 91–96.

15. M. Valdivia, *Absolutely convex sets in barrelled spaces*, Ann. Inst. Fourier (Grenoble) **21**, (1971), fasc. 2, 3–13.

16. J. H. Webb, *Sequential convergence in locally convex spaces*, Proc. Cambridge Philos. Soc. **64** (1968), 341–364.

17. J. H. Webb, *Countable-codimensional subspaces of locally convex spaces*, Proc. Edinburgh Math. Soc. **18** (1972/73), 167–172.

18. Y.-C. Wong, *Order-infrabarrelled Riesz spaces*, Math. Ann. **183** (1969), 17–32.

19. Y.-C. Wong and K.-F. Ng, *Partially ordered topological vector spaces*, Oxford Math. Monograph, Clarendon Press, Oxford, 1973.

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