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Monotone operators and applications in control and network theory, by V. Dolezal, Studies in Automation and Control 2, Elsevier, Amsterdam, The Netherlands, 1979, x + 174 pp., $43.50.

A decade ago the term "applied mathematics" was synonymous with mathematical physics and fluid mechanics. The tools of the "applied mathematician" were differential and integral equations, integral transforms, numerical analysis and a touch of optimization theory. If someone wanted to apply algebraic or differential geometry, several complex variables, operator theory, functional analysis or any of the several fields of algebra, it wasn't "applied mathematics". Of course, it wasn't pure mathematics either, nor was it science or engineering. Fortunately, during the past decade the applied mathematician has come in out of the cold with the emergence of the mathematical theory of networks and systems in which each of the above topics has found significant applications.

Although the field has evolved far beyond the point of exposition in a single volume,[1] Dolezal's text is representative of the operator theoretic approach to the mathematical theory of networks and systems [2], [8], [12], [13]. Intuitively, a system is a "black box" whose inputs and outputs are functions of time (or vectors of such function). As such, a natural model for the system is an operator defined on a function space. This observation and its corollary to the effect that system theory is a subset of operator theory, unfortunately, proved to be the downfall of the early researchers in the field. The projection theorem was used to construct optimal controllers which proved to be unstable, operator factorizations were used to construct stochastic filters which were unrealizable, and unitary extension theory was used to construct lossless networks which were noncausal.

The difficulty lies with the fact that the operators encountered in system theory are defined on spaces of time functions and, as such, must satisfy a physical realizability (or casuality) condition to the effect that the operator cannot predict the future. Although this realizability condition usually takes care of itself in the analysis problems of classical "applied mathematics" it must be externally imposed on the synthesis problems which play the key role in system theory. Indeed, since the causal operators are not closed under inversion or adjoints, even if one begins with realizable system specifications, the system synthesized therefrom may fail to be realizable if either an adjoint or an inverse is employed in the derivation.

If F and G are spaces of functions defined on a linearly ordered time set, we say that $W: F \to G$ is causal (realizable) if whenever

$$f_1(t) = f_2(t), \qquad t \leqslant s, \tag{1}$$

then

$$[Wf_1](t) = [Wf_2](t), \qquad t \leqslant s. \tag{2}$$

[1]For a review of the overall field see any of the three recent journal special issues in the area [5], [6], [10].

This implication corresponds to the usual physical interpretation of causality to the effect that W cannot predict the future. Moreover, it characterizes convolutional kernels with support on the half line, lower triangular Toeplitz matrices, Volterra operators, and multiplication by a function in H^∞ and is thus no stranger to the mathematician. Unfortunately, this simple function space definition does not abstract to the topological vector spaces commonly used in operator theory wherein the concept is undefined. As such, even though a system may be naturally modeled as an operator on a function space the classical theorems of operator theory are incapable of coping with the realizability conditions required for a meaningful system theory and have thus had little impact on the subject.

This realizability problem can be overcome by incorporating a resolution of the identity into the topological vector space on which the system is defined [2], [12], [13], which, in turn, allows one to define a causal system (operator). Indeed, once this is achieved one discovers that the causal operators on a Hilbert space form a nest algebra [11] which is invariant on an appropriate chain of subspaces. As such, both the theory of nest algebras [11] and the theory of triangular operators [1], [3] may be invoked in support of the system theorist. With this observation the operator theoretic approach to the theory of networks and systems was off and running and a number of researchers set out to formulate a unified theory of linear networks and systems via operator theoretic techniques. At the present time this theory is reaching maturity [5], [6], [10] and much to the surprise of the early practitioners of the art (including this reviewer) a considerable theory of nonlinear networks and systems has been developed along the way [8], [13].

Dolezal's text represents the first exposition in book form of the resultant nonlinear system theory. The text is built around three major chapters dealing with monotone operators, the analysis of feedback systems, and the analysis of electric networks. By a monotone operator, W, we refer to a (possibly multivalued) operator defined on a Hilbert space such that

$$\langle Wf_1 - Wf_2, f_1 - f_2\rangle \geqslant 0 \tag{3}$$

for all f_1 and f_2 in the domain of W. As such, the concept may loosely be viewed as an extension of the class of operators with positive derivative. With respect to the present application, such an operator is a natural model for a passive electric network and the slope restricted nonlinearities commonly used in control theory [13]. Of course, one can define a number of variations on the theme obtaining strictly or strongly monotone operators, causal monotone operators, etc. Each of these classes is formulated in the text and their fundamental properties delineated.

In the system theory literature a feedback system is typically represented by a block diagram such as that shown below. Here the black box denoted by P represents a plant; an aircraft, an electric generator, a chemical process, etc., which is to be controlled by the compensator, C. Typically, the plant is a large piece of hardware whose characteristics have been predetermined and it is the responsibility of the systems engineer to design a compensator which will make the plant behave in a prescribed manner. To this end the engineer has considerable control over the design of the compensator, C, which may be implemented in hardware or simply as a program for a control computer. Whatever form it takes the compensator observes the difference between the reference input to the system, r, and the plant output, y, and uses this data to determine the plant input. For instance, r may represent the desired route for an aircraft and y the actual route in which case C computes a control input (to the elevator, flaps, rudder, etc.) which will compensate for any deviation between the desired and actual route.

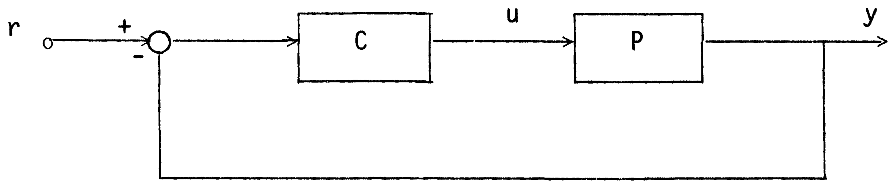

FIGURE 1. Typical block diagram.

From a mathematical point of view the block diagram of Figure 1 can be summarized by the pair of equations

$$y = Pu \tag{4}$$

and

$$u = C(r - y) \tag{5}$$

where r, u, and y lie in a Hilbert (Banach) space, H, and P and C are operators on H.[2] The basic questions of feedback system analysis then reduce to

(i) When does the feedback system (i.e. equations (4) and (5)) define a mapping T such that $y = Tr$?

(ii) When T is well defined what are its properties? Is it stable? Is $\partial T/\partial P$ sufficiently small? etc.

Indeed, the answers to these questions and the converse question of designing a compensator, C, for a given plant, P, which will cause T to have certain prescribed properties, are the essence of feedback theory. The present text resolves many of the basic problems of feedback system analysis in the case where P and/or C are monotone operators. Dolezal, however, makes only a minimal attempt to go beyond the monotone case nor does he even formulate, let alone resolve, the much deeper feedback system design problem.

Although an electric network is typically represented by a schematic diagram which is quite unlike the block diagram of Figure 1 and Dolezal's mathematical abstraction is equally unlike equations (4) and (5), the mathematical theory of networks is, in fact, quite similar to the above described feedback system theory. Following Dolezal an abstract network is an ordered triple $[Z, N, M]$ where Z is a (possibly multivalued) operator on a Hilbert space, H, and M and N are complementary subspaces in H. Physically, M and N represent spaces of admissible voltage and current vectors, respectively (i.e., vectors satisfying appropriate generalizations of the Kirchhoff voltage

[2]More generally, we may take r and y to be in H^1 and u to be in H^2 with C mapping H^1 to H^2 and P mapping H^2 to H^1.

and current laws), while Z represents a component impedance operator which is passive if Z is monotone. Given a vector $e \in H$, a vector $i \in N$ is termed a solution of the network if there exists $v \in H$ such that $v = Zi$ and $v - e \in M$.

Given the above definition one may formulate and resolve a number of problems in abstract network theory.

(i) When does an abstract network have a solution?

(ii) When an abstract network admits a solution, what are its properties? Is it causal? Can it be viably approximated by a linear network? etc.

With the aid of appropriate monotonicity assumptions on Z most of the above questions are resolved in the text. Moreover, it is shown that the resultant abstract network theory subsumes the classical network analysis problem and most of the infinite network theories which have been proposed during the past decade [4], [14]. As such, Dolezal's chapter on Monotone Networks represents a significant contribution to the literature on passive network analysis.

In addition to its three major chapters the text contains four short chapters in which certain technical questions are resolved. Most importantly, Chapter 3 develops the theory of operators defined on an extended Hilbert space [13]. Like the introduction of the resolution structure which allowed the concept of causality to be formalized, the development of the extended space concept was one of the key contributions which made possible today's mathematical theory of networks and systems. Simply stated the early researchers in the field were faced with a fundamental dilemma. If their system theory was to take advantage of the existing theory of operators it necessarily must be formulated in a normed space. On the other hand, if unstable systems were to be included in the theory unbounded signals must be allowed. To achieve the best of both worlds, the given normed space on which the system was defined was embedded in an appropriately constructed extension space which included unbouned signals. Moreover, the extension was constructed in such a way that any causal operator on the normed space could be uniquely lifted to the extension space [13]. This, in turn, allowed for the possibility of unstable systems in a system theory defined on a normed space.

In summary, the present text represents the first attempt at an exposition of the operator theoretic approach to system theory to emphasize nonlinear systems. Although the exposition on the feedback systems is incomplete the exposition on passive network analysis is excellent. Moreover, the text develops most of the fundamental tools and techniques which are required by a researcher in the field. As such, it represents an ideal starting point for someone interested in initiating a research program in the area. Since the use of advanced operator theoretic techniques is kept to a minimum the text is suitable for a graduate course for students (in mathematics or engineering) with a real analysis background.

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R. SAEKS

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Gaussian random processes, by I. A. Ibragimov and Yu. A. Rozanov, Applications of Math., volume 9, Springer-Verlag, New York-Heidelberg-Berlin, 1978, x + 276 pp., $24.80.

A Gaussian law (= probability measure) P on a finite-dimensional vector space V is of the form $dP(x) = \exp(-Q(x))\, dx_J$, where Q is a quadratic polynomial and dx_J is Lebesgue measure on a linear variety (affine subspace) J. Such laws, also called *normal*, are staples of multivariate statistics ([1], [34], [43]), along with their relatives such as Wishart distributions.

Let $EX = \int X\, dP$, the mean of the (vector or scalar) X. In the rest of this review *Gaussian laws will all have mean* 0 unless otherwise stated. If A, B, C and D are any four linear forms on V, then $E(ABCD) = E(AB)E(CD) + E(AC)E(BD) + E(AD)E(BC)$. So, $E(A^4) = 3E(A^2)^2$, the first of a sequence of identities which characterize Gaussian laws on $\mathbf{R}^1$.

Given a probability space $(\Omega, \mathfrak{B}, \Pr)$ and any set T, a *Gaussian process* is any real function X on $T \times \Omega$ such that for each finite set $F \subset T$, $\{X(t, \cdot)\}_{t \in F}$ has a Gaussian law on $\mathbf{R}^F$. Let $X(t) \equiv X(t, \cdot)$.

If T is a Hilbert space H, the *isonormal* Gaussian process L maps H isometrically into an $L^2(\Omega, \Pr)$, with $EL(x, \cdot)L(y, \cdot) = (x, y)$, the inner product; this fixes the laws of L. For any Gaussian process X, there is a Y with the same laws and $Y(t, \omega) = L(g(t), \omega)$, where g maps T into some Hilbert space H. So L is *the* Gaussian process [13]; it clothes a pristine Hilbert space in full Gaussian attire.

Trajectories. Probabilists like to pick an ω and follow the wandering path, or sample function, $t \to X(t, \omega)$ ([3], [13], [20], [48]). The speed at which $\exp(-x^2/2)$ goes to 0 as $x \to \infty$ lets us make (almost) all paths continuous if $g(T)$ in H is compact enough. If $T = \mathbf{R}$, the process X is called *stationary* if all its laws are preserved by translations $t \to t + h$. For a stationary X