47. M. A. Piech, *Smooth functions on Banach spaces*, J. Math. Anal. Appl. **57** (1977), 56–67.
48. V. I. Piterbarg, *Some directions in the investigation of properties of trajectories of Gaussian random functions: Supplement-survey*, Stochastic Processes: Sample Functions and Intersections, Matematika: novoe v zarubezhnoĭ nauke, vol. 10, Mir, Moscow, 1978, pp. 258–280. (Russian)
49. L. D. Pitt, *A Gaussian correlation inequality for symmetric convex sets*, Ann. Probability **5** (1977), 470–474.
50. M. Reed and L. Rosen, *Support properties of the free measure for Boson fields*, Comm. Math. Phys. **26** (1974), 123–132.
51. I. E. Segal, *Tensor algebras over Hilbert spaces*, Trans. Amer. Math. Soc. **81** (1956), 106–134.
52. ______, *Distributions in Hilbert space and canonical systems of operators*, Trans. Amer. Math. Soc. **88** (1958), 12–41.
53. L. A. Shepp, *Radon-Nikodym derivatives of Gaussian measures*, Ann. Math. Statist. **37** (1966), 321–354; correction, Ann. Probability **5** (1977), 315–317.
54. D. Slepian, *The one-sided barrier problem for Gaussian noise*, Bell System Tech. J. **41** (1962), 463–501.
55. V. N. Sudakov, *Geometrical problems in the theory of infinite-dimensional probability distributions*, Trudy Mat. Inst. Steklov. **141** (1976), Nauka, Leningrad (Russian); English transl., Proc. Steklov Inst. Math. (1979).
56. N. Wiener, *The homogeneous chaos*, Amer. J. Math. **60** (1939), 897–936.
57. E. Wong, *Recent progress in stochastic processes: a survey*, IEEE Trans. Information Theory **IT19** (1973), 262–275.

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Foundations of mechanics, Second Edition, Revised and enlarged, by Ralph Abraham and Jerrold E. Marsden, The Benjamin/Cummings Publishing Company, Reading, Mass., 1978, xxii + 806 pp., $39.50.

1. This excellent book is one of several superb books on mechanics which have appeared in the past decade, such as those of Souriau [**10**], Siegel-Moser [**9**], Arnold [**2**] and Thirring [**13**], indicating a revitalized interest in the venerable subject of classical mechanics. Actually, there have been at least three sources of revitalization in the past forty years. The first came from the solution of the "small divisor problem" in celestial mechanics. The breakthrough here was achieved by Siegel in a mathematical tour de force, and then a new powerful general principle was discovered by Kolmogorov and developed in the hands of Arnold and Moser into a major analytical tool. The second came from the study of geometric properties of mappings and flows, especially in their "generic" behavior. The guiding philosophy had come from the foundational work in differential topology of Whitney and Thom, and was developed by Smale, Anosov, Sinai and their schools. More recently, there has been an influx of new ideas coming from group theory, from the work of Kirilov and Kostant in representation theory, and of Souriau, in rethinking the physical and geometrical principles underlying mechanics. As the bulk of the material added in the second edition deals with this last topic, I will concentrate my attention on it.

Much, but not enough, has been written about the philosophical problems involved in the application of mathematics, and particularly of group theory, to physics. There is the celebrated paper of Wigner [15] with whose thoughts I wholeheartedly agree; yet for all its eloquence it leaves much unsaid. On the one hand, mathematics is created to solve specific problems arising in physics; on the other, the mathematics provides the very language in which the laws of physics are formulated. One need only think of calculus or of Fourier analysis as examples of this dual relationship. A third example is provided by Hamiltonian mechanics. We are all familiar with the exploitation of symmetry in the solution of a mathematical problem. On the other hand, the very assertion of symmetry is often the profoundest formulation of a physical law, or the key step in the development of a new theory.

The mathematical theory underlying Hamiltonian mechanics is currently called symplectic geometry. We briefly recall the basic definitions and the early history. A symplectic vector space is a real vector space equipped with an antisymmetric, nondegenerate bilinear form. For example, on $\mathbf{R}^2$ we can define the form (,) by $(u_1, u_2) = q_1p_2 - q_2p_1$ where $u_1 = \binom{q_1}{p_1}$ and $u_2 = \binom{q_2}{p_2}$. It is not hard to see that such a vector space must be even dimensional (if finite dimensional). A linear isomorphism of a symplectic vector space, V (or more generally of V onto W), is called symplectic if it preserves the bilinear form. (In two dimensions, a linear transformation is symplectic if and only if its determinant is one. In higher dimensions, the condition is more restrictive.) A differentiable map of an open subset of a symplectic vector space V into V is called symplectic if its (Jacobian) matrix of partial derivatives is symplectic at every point. A symplectic manifold M is an even-dimensional manifold which locally has the structure of a symplectic vector space. This means that one has local charts ψ_i mapping open sets U_i of M onto open subsets of some fixed symplectic vector space V, and such that the change of coordinates maps $\psi_i \circ \psi_j^{-1}$ (defined on $\psi_j(U_i \cap U_j)$) are symplectic. (Alternatively, thanks to a theorem of Darboux, a symplectic manifold is a manifold together with a closed two form of maximal rank.) One has the obvious definition of symplectic diffeomorphisms (i.e. one-to-one smooth transformations with smooth inverse, which are locally symplectic in the above sense). In the older literature, symplectic diffeomorphisms were called canonical transformations. Symplectic geometry is the study of symplectic manifolds and diffeomorphisms. The relation with mechanics is usually expressed by saying that the "phase space" of a mechanical system is a symplectic manifold, and the time evolution of a (conservative) dynamical system is a one parameter family of symplectic diffeomorphisms. The role of the symplectic structure had first appeared, at least implicitly, in Lagrange's work on the variation of the orbital parameters of the planets in celestial mechanics. But its central importance emerged from the work of Hamilton.

At the age of eighteen, Hamilton submitted a paper entitled "Caustics" to Dr. John Brinkley, then the first royal astronomer for Ireland, who, as a result, is said to have remarked "This young man, I do not say *will be* but *is* the first mathematician of his age". Brinkley presented the paper to the Royal Irish Academy. It was referred as usual to a committee whose report, while acknowledging the novelty and value of its contents, recommended that it should be further developed and simplified before publication. Five years later, in greatly expanded form, the paper finally appeared, entitled "Theory of systems of rays" published in 1828 in the Transactions of the Royal Irish Academy. The gist of Hamiltonian optics, in modern language, is as follows: One is interested in studying the geometry of rays of light as they pass through some optical system. Suppose our system is aligned along some axis, and we study rays which enter the system at the left and emerge from the right. The portion of the rays to the left of the system are straight line segments. One needs four variables, locally, to specify such a line–two variables to specify the point of intersection of the line with a plane perpendicular to the optical axis, and two additional angular variables giving the inclination of the line to this plane. The problem is to relate the incoming line segments to the left of the system to the outgoing line segments to the right. The first basic assertion is that if we use the right coordinates (which involve the index of refraction of the ambient space) the transformation from the incoming to the outgoing coordinates is a symplectic diffeomorphism. Thus geometrical optics is reduced to symplectic geometry. Hamilton shows that if the graph of a symplectic transformation satisfies an appropriate transversality condition, then the transformation determines and is determined by a function of half the incoming and outgoing variables, the so called generating function of the symplectic transformation. As this function is determined solely by the physical properties of the optical system, Hamilton calls it a characteristic function. Depending on the transversality assumptions made, it can be a function of the points of intersection of the incoming and outgoing rays with the transversal planes–the point characteristic, the incoming points of intersection and the outgoing angles–the mixed characteristic, or the incoming and outgoing angles–the angle characteristic. These functions are of use in combining optical systems, i.e. in composing the corresponding symplectic transformations. They are also extremely useful in describing the deviation of the symplectic transformations from linearity–the "geometric aberrations" of the optical system. Finally, they are closely related to the "optical length" of the light rays themselves, and these light rays can be characterized as being extremals for optical length–"Fermat's principle". See [**4**, Chapter III], for a modern discussion of these ideas and their applications in optics. Some years later Hamilton realized that this same method applies unchanged to mechanics: replace the optical axis by the time axis, the light rays by the trajectories of the system, and the four incoming and four outgoing variables by the $2n$ incoming and outgoing variables of the phase space of the mechanical system. Hamilton's methods, as developed by Jacobi and other great nineteenth century mathematicians, became a powerful tool in the solution or analysis of mechanical problems. Hamilton's analogy between optics and mechanics served as a beacon to the development of quantum mechanics some one hundred years later.

One of the key new ideas that Hamilton introduced into mechanics is that one must consider the most general symplectic transformations of phase space. Lagrange had already introduced the notion of "generalized coordinates". In modern terminology this means that one should consider a possible configuration of a mechanical system as a point in a differentiable manifold, and, of course, then use any system of local coordinates which are convenient. The corresponding phase space is currently called the "cotangent bundle" and is denoted by T^*M where M is the manifold. It carries a natural symplectic structure. If $q = (q_1, \ldots, q_n)$ are local coordinates on an open set U of M, then $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ are local coordinates sitting over U in T^*M and provide a symplectic chart. Lagrange allows any change of the q coordinates (which then induce a change in the p coordinates as well). Hamilton allows any symplectic coordinates on T^*M, a coordinate change that might mix the q's and the p's. It is a consequence (a version really) of Darboux's theorem that any symplectic manifold looks locally like a T^*M. It is rather surprising that it is only in the past ten years that the next small step was seriously considered, that one should use general symplectic manifolds, which are not necessarily globally cotangent bundles, in the formulation of mechanics. This step was suggested by Bacry, Kostant and others, but first fully developed with its physical implications by Souriau.

To give a taste of how group theory, combined with symplectic geometry, can give some physical insight, let us return to Hamilton's optics. The incoming or the outgoing data, or the data at any plane of the optical system, are ways of parametrizing the light rays. The fact that the transformation from one set of data to another is symplectic can be reformulated as saying that the space of light rays of an optical system carries a natural symplectic structure. (This is the point of view that Souriau consistently maintains towards mechanics.) Let us consider the simplest possible optical system–one with no lenses, no varying index of refraction, just empty space with a constant index of refraction. The light rays are then straight lines. The space of straight lines in Euclidean three space is a four-dimensional manifold, upon which the group of Euclidean motions acts transitively. The symplectic structure should be invariant under the Euclidean group. So we can pose the following question: what are the possible symplectic structures on the space of lines which are Euclidean invariant. Thanks to the work of Kirilov, Kostant, and Souriau, this kind of question has a completely general answer. Given any Lie group, there is a standard procedure for finding all the symplectic manifolds on which the group acts transitively. If the group satisfies a certain mild cohomological restriction, and all the Euclidean groups in three or more dimensions do, the answer is as follows. Let G be the group and g its Lie algebra. The group G acts on g via the adjoint representation. (Think of G as a group of matrices and g as the space of those matrices X such that $\exp tX$ is in G for all t. Then $A \in G$ acts on $X \in g$ by sending it into AXA^{-1}.) If Ad_A denotes the transformation of g corresponding to A, and if $\mu \in g^*$ is a linear function on g, define $A \cdot \mu$ by $(A \cdot \mu)(X) = \mu(\mathrm{Ad}_{A^{-1}}X)$. In the language of group theory, this defines the contragredient representation on the dual space g^* of g. Let $G \cdot \mu = \{A \cdot \mu, A \in G\}$ be the G orbit through μ. Then this orbit, $G \cdot \mu$, carries a natural, G invariant, symplectic structure. Furthermore, these are all the transitive symplectic G spaces in the sense that any symplectic manifold on which G acts transitively is a covering space of some such orbit by a covering map which is locally a symplectic diffeomorphism. For semidirect products such as the Euclidean group, there is a simple recipe for describing the orbits, cf. **[12]** or **[3**, Chapter IV, §7**]**. Applied to the three-dimensional Euclidean group, the answer is as follows: There is a two parameter family of four-dimensional orbits, a one parameter family of two-dimensional orbits, and a single zero-dimensional orbit (the origin). Each of the four-dimensional orbits is equivalent, as a G space, to the space of straight lines; they differ in their symplectic structures. In other words, there is a two parameter family of invariant symplectic structures on the space of straight lines. Let us describe the parametrization of these structures: Fix a straight line, say the z-axis. The subgroup, H, which fixes this axis consists of rotations about, followed by tıanslations along, this axis. We must find all μ which are left invariant by this subgroup H. Let h denote the Lie algebra of H. A basis of H is given by R_z and T_z where R_z denotes infinitesimal rotation about the z-axis and T_z denotes infinitesimal translation along the z-axis. It is easy to see that if μ is left invariant by H, it must vanish when evaluated on infinitesimal rotations or translations about the x or y axes. Thus μ is determined by its values

$$\mu(R_z) = s$$

and

$$\mu(T_z) = r.$$

We may fix R_z so that $\exp 2\pi R_z = 1$ and thus s is a real number; since we may rotate the positive z-axis into the negative, we see that s can be taken to be a nonnegative real number. A choice of T_z is equivalent to a choice of unit of length, so we see that r has the units of inverse length. In fact, the value $r = 0$ is not allowed, for it turns out that this corresponds to a two-dimensional orbit. So we may write $r = \lambda^{-1}$ where λ has the units of length. What is the physical significance of this length? The linear function μ can be restricted to h, i.e. thought of as a linear function on h. It vanishes on the commutator subalgebra $[h, h]$–here because h is abelian, but the corresponding fact is true in general. We can think of μ as an "infinitesimal character" on h, and try to define the corresponding character χ_μ on H by

$$\chi_\mu(\exp X) = e^{2\pi i\mu(X)}$$

for $X \in h$. There will be trouble at those $X \in h$ for which $\exp X = id$, and the definition only applies to the connected component of H, so further specification would be required if H were not connected (which would occur in our case if the Euclidean group included reflections). For the translation subgroup, T, of H these problems do not arise, and we can write

$$\chi_\mu(d) = e^{2\pi i d/\lambda}$$

where d stands for translation through distance d along the z-axis. Let us compare this with the wave theory of light as developed by Young and Fresnel. Newton, in his famous experiments with the prism, had come to the conclusion that "colours are original and connate properties of light", and that "to the same degree of refrangibility ever belongs the same colour and to the same colour ever belongs the same degree of refrangibility". That white light is really a superposition of the more elementary light rays of specific color. Young, in the Philosophical Transactions for 1802, writes of his discovery of a "simple and general law" that "wherever two portions of the same light arrive at the eye by different routes, either exactly or very nearly in the same direction, the light becomes most intense where the difference of the routes is a simple multiple of a certain length, and least intense in the intermediate state of the interfering portions; and this length is different for light of different colours". Thus each "elementary" kind of light is associated with a definite length. Fresnel explained the phenomenon of interference of monochromatic light as follows (we paraphrase his explanation): Associated with each point on a light ray of geometrical optics is a complex number, c (determined by an overall phase factor). As the light propagates along the ray, not only does the amplitude, $|c|$, decrease due to the attenuation of the light, but the phase changes; the change in phase as we move along a ray a distance of optical length d is given by multiplication by the factor $e^{2\pi id/\lambda}$ where λ is defined to be the wave length of the light. If light arrives at some point from various rays, we add the c's from each ray to obtain a total value, C. The intensity of the light at the point is proportional to $|C|^2$. This formulation of Fresnel, when combined with geometrical optics, is sufficient to explain the observed phenomena of interference and diffraction. But we see that our parameter λ is nothing other than the wave length of the light! If we ignore the attenuation factor, we can formulate Fresnel's analysis of the phase change in more modern language by saying that light is associated with a complex line bundle over the space of rays of geometrical optics. For a homogeneous medium, where the light rays are straight lines, any homogeneous such bundle will be specified by a character, χ, on the subgroup, H, fixing a single line. Restricted to the translation subgroup, T, of H, we must have $\chi(d) = e^{2\pi id/\lambda}$ for some λ (unless $\chi(d) \equiv 1$) and this λ is the wave length of the light. What about the parameter s? In the Euclidean group, $\exp 2\pi R_z = id$. Since $1 = \chi(id)$ for any character, and $\chi_\mu(2\pi R_z) = e^{2\pi is}$, we see that s must be an integer. (Actually, as in more modern theories, we might want to consider the universal covering group of the Euclidean group, which is a double cover, and in which $\exp(2\pi R_z) \neq id$ but $\exp(4\pi R_z) = id$. Then the condition becomes that $2s$ is an integer.) Does the integer s make its appearance in the theory of light? Again the answer comes from the work of Young and Fresnel, this time on polarized light. To explain the phenomenon of polarization, they were led to conclude that light consists of "transverse vibrations". In terms of the preceding description we must take c not to be a complex number but a complex two-dimensional vector normal to the light ray. In more modern language, we must look at the complexified normal bundle, which induces a two plane bundle on the space of lines. The two-dimensional complex representation of the group of rotations about the z-axis splits into a direct sum of two one-dimensional representations; and the associated line bundles correspond to $s = 1$ and $s = -1$, corresponding to right-handed and left-handed circularly polarized light. It is interesting to read the papers of Arago and Fresnel and see how much Euclidean geometry enters into their discussion. The idea that, in general, one should single out certain orbits corresponding to μ's satisfying integrality conditions, and that these are associated with complex line bundles, is known as prequantization. It was introduced and developed by Kostant in his fundamental paper **[6]** and is a key step in "geometrical quantization".

So far I have described some actual history of physics, albeit from a special point of view. Now I want to engage in some historical science fiction. Suppose that mechanics had developed before the invention of clocks. So we could observe the trajectories of particles, their collisions and deflections, but not their velocities. For instance we might be able to observe tracks in a bubble chamber or on a photographic plate. (In the case of light, all of the work described above was done before there was any accurate measurement of the velocity of light.) The configuration space of a single particle is just the three-dimensional Euclidean space, E^3, the corresponding phase space, T^*E^3 is six dimensional, with coordinates (q, p) where q and p are three vectors. The Euclidean group acts on T^*E^3. A rotation A sends (q, p) into (Aq, Ap) while translation through vector v sends (q, p) into $(q + v, p)$. These are clearly symplectic transformations as required. The Euclidean group does not act transitively on T^*E, since $\|p\|$, called the total momentum is invariant. The collection of all (q, p) with a constant value of $\|p\|$ is a five-dimensional manifold on which the Euclidean group acts transitively. The points $(q + tp, p)$ all lie on a straight line, and it follows from some elementary symplectic geometry that the symplectic structure of T^*E, together with the choice of $\|p\|$, determines a symplectic structure on the space of lines. Since this symplectic structure is invariant under the Euclidean group, it must coincide with one of those described above. In fact, an easy computation shows that it is the one with $s = 0$ and $r = \|p\|$. Thus each "free particle" is parametrized by its total momentum. (In the absence of the notion of velocity, we cannot distinguish between a "light particle moving fast" or a "heavy particle moving slowly". Of course, from the scattering experiments themselves, we would be led to discover new conserved quantities such as energy, and thus be led to enlarge the group. But I do not want to go into this point.) Without some way of relating momentum to length, we would introduce "independent units" of momentum, perhaps by combining particles in various ways and performing collision experiments. But we know that the "natural units" should be inverse length. A single experiment, the photoelectric effect, involving an interaction between light and one of our "particles" would then give us the conversion factor, and allow us to write $\|p\| = h/\lambda$. Thus, from this group theoretical point of view, Planck's constant, h, is a conversion factor from the "independent" units of momentum to the "natural" units of inverse length. Of course, the story did not develop that way. The "conversion factor" was first found between "energy" and "inverse time"; but to explain this would involve us in larger groups such as the Galilean or Poincaré groups and take us too far afield.

Group theory, in conjunction with symplectic geometry, is a powerful tool in the solution of the equations of motion of certain mechanical systems. Suppose that the group G acts on some symplectic manifold, N, as symplectic diffeomorphisms. Under certain additional hypotheses, one can conclude the existence of a map $\Phi: N \to g^*$ such that $\Phi(An) = A \cdot \Phi(n)$ for all $A \in G$ and $n \in N$. This map is the group theoretical generalization of the notion of (total) linear or angular momentum. It was discovered in this generality by Souriau and called the moment map. In a fundamental paper [8], Marsden and Weinsten show that (under suitable technical hypotheses) $\Phi^{-1}(\mu)$ is naturally associated with an auxiliary ("reduced") symplectic manifold, $S_\mu = \Phi^{-1}(\mu)/G_\mu$ where G_μ is the isotropy group of μ. If the Hamiltonian, H, is invariant under G, then the Hamiltonian flow leaves $\Phi^{-1}(\mu)$ invariant; the Hamiltonian determines a function, H_μ on S_μ, and this reduced Hamiltonian system, if solved, can be used to reconstruct the flow on $\Phi^{-1}(\mu)$. All of this is very clearly explained in Abraham and Marsden, §4.3. In [5] this procedure is reversed, and a complicated looking flow on a certain S_μ is shown to be equivalent to a simple looking one on N. Also in [5] the structure of the inverse image of an entire orbit is investigated. If H, on the other hand, is of the form $H = P \circ \Phi$ where P is some function on g^*, then the study of the flow on N can be reduced to the study of the flow given by P on various orbits, cf. [4]. In fact, cf. [7], various classical and modern mechanical systems can be identified and solved as flows on orbits with Hamiltonians having a group theoretical significance. Indeed, thanks to the efforts principally of Kostant and Moser, all the classically integrable systems save one have a nice group theoretical interpretation. The one classical system, which at the time of this writing is still not "understood" from the group theoretical viewpoint, is Kowalewskaya's top–the motion of a symmetric rigid body about a fixed point, when two of the principal moments of inertia at the fixed point are equal and are double the thrd, with the center of gravity situated in the plane of the equal moments of inertia.

2. Let me now turn to an analysis of the book. The development of the ideas described above requires a considerable amount of differential geometry. The first part of the book, 156 pages, is devoted to a self-contained treatment of all the necessary geometrical tools. The authors have done a first rate job, and I know of no better source from which to learn this material. I have tried one or two expositions of this sort of stuff myself, so I am very familiar with the pedagogical difficulties that the authors have overcome. Part II is entitled "Analytic Dynamics", consisting of three chapters. The first of these gives a clear, more or less standard, presentation of Lagrangian dynamics on the tangent bundle, Hamiltonian mechanics on the cotangent bundle, the variational calculus and the relations among them. The next chapter deals with mechanical systems with symmetry. It deals with Lie group actions on manifolds, the moment map, the Kirilov-Kostant-Souriau theorem, the reduction of systems with symmetry (as mentioned above), Smale's program for the study of Hamiltonian systems with symmetry, all of these very well presented, and the rigid body. The treatment of the rigid body follows Arnold's paper [1] which was among the first group theoretical treatments of mechanics in the modern literature. It is indeed pleasant to learn that the basic theorem of Poinsot on the motion of the rigid body without forces–that "the polhode rolls along the herpolhode without slipping in the invariable plane" is valid for the left invariant geodesic flow on any Lie group. But does one really need six different formulations of the Euler conservation laws in a theorem whose statement extends over two pages? More seriously, there is no discussion whatsoever of the rigid body in the presence of forces. Lagrange's top–a rigid body with two moments of inertia equal and with the center of mass along the third axis in a uniform gravitational field–can be given a completely group theoretical treatment. One should compare Abraham and Marsden's discussion with the treatment in Arnold's book, where, in a few short pages, one gets a clear picture of the motion. The third chapter in this part deals with the Hamilton-Jacobi theory, infinite dimensional Hamiltonian systems, an introduction to nonlinear oscillations and quantization. The proof of von Hove's theorem on the impossibility of complete quantization is very clearly presented and very illuminating. The remaining topics are also presented with great care and at a high level of exposition. Throughout these three chapters, an enormous amount of additional material is presented in the form of exercises, making the book indispensable as a reference. The third part, "An Outline of Qualitative Dynamics" is just that. Many results are stated without proof, referring the reader to other sources. A comprehensive survey is given of topological dynamics, differentiable dynamics and a qualitative discussion of Hamiltonian dynamics. A very instructive collection of figures of stable bifurcations of vector fields, with exotic names attached and of Hamiltonian vector fields (in dim. 4) are presented. The literary style here is also refreshingly different. One gets the impression of hearing a sports fan relive some favorite moment. Thus, on page 535, we read about a proposed definition that "Within days of the proposal of these weaker notions of stability, the first of the counterexamples was constructed, killing hopes that they . . . ". Sometimes the authors get carried away in group rhetoric. Thus, on page 542, we read "What of the future? The gap $A \subset\subset G_4$ still prevents pilgrims from climbing to heaven by performing good works."

The fourth part is called "Celestial Mechanics". A very careful discussion is given of various models of the Kepler problem, and of the restricted three body problem, perhaps overly detailed. (With understandable prejudice, I prefer the treatment in my book **[11]**, which although a bit sloppy from a mathematical point of view, gets to the desired goal much faster.) No proof of the basic implicit function theorem is given; the reader is referred to other sources, presumably either my book **[11]** or Siegel-Moser **[9]**. More seriously, no mention is given of some of the more important recent developments in this direction, such as the remarkable solution by M. Herman of the Poincaré circle problem, and the applications to the study of groups of diffeomophisms by Thurston, Mather, Banyaga, and others. The book closes with a clean treatment of the work of Smale and Iacob on the topology of the planar n body problem. All in all, a remarkable achievement of scholarship and exposition.

Perhaps some comments are in order comparing Abraham and Marsden with some of the other recent books. First of all, Abraham and Marsden is a book on mathematics, not physics. It has as its goal the elucidation of the mathematical structures underlying mechanics and areas of pure mathematics that have been stimulated by problems in mechanics. It does not try to use the mathematical structures as organizing principles for physical insight. Thirring **[13]** is a book on physics, indeed part of a lecture course series on theoretical physics. Thirring does not have any discussion of the interaction of group theory with mechanics, or of qualitative dynamics in the broad sense, two major themes in Abraham and Marsden. Thirring does always have his eye on broader physical issues. Thirring is succinct, Abraham and Marsden is encyclopedic. Souriau [10] is a deep and original book. At every turn one finds novel profound insights into the relation between geometry and physics, not only in mechanics but also in statistical mechanics and quantum mechanics. It has already had a major impact on portions of the mathematical community; one hopes that it will have a similar influence on physicists. Unfortunately, its eccentric notation probably makes it impenetrable to most. It is not easy reading. Arnold [2] is a pleasure to read from beginning to end. Written by the sure hand of a master, every detail shines like a polished jewel, and the overall structure has the strength of a coherent point of view. It is primarily a book on mathematics but the interactions with physics are not neglected. I have not taught from it, but imagine that it makes the perfect text. Siegel-Moser [9] is also a classic, written by masters of the subject. It is more restricted in scope, being concerned with the deep mathematical theorems of celestial mechanics. All the above books should be on the shelf of every serious student of mechanics. One would like to be able to report that such a collection would be complete. Unfortunately, this is not so. There exist topics in the classical repertoire, such as Kowalewskaya's top which are not covered by any of these books. So hold on to your copy of Whittaker [14].

REFERENCES

1. V. I. Arnold, *Sur la géometrie différentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier (Grenoble) **16** (1966), 319–361.

2. ______, *Mathematical methods of classical mechanics*, Graduate Texts in Math., no. 60, Springer-Verlag, Berlin and New York, 1978.

3. V. Guillemin and Shlomo Sternberg, *Geometric asymptotics*, Amer. Math. Surveys, no. 14, Amer. Math. Soc., Providence, R. I., 1977.

4. ______, *The moment map and collective motion*, Ann. Phys. (to appear).

5. D. Kazhdan, B. Kostant and S. Sternberg, *Hamiltonian group actions and dynamical systems of Calogero type*, Comm. Pure Appl. Math. **31** (1978), 481–508.

6. B. Kostant, *Quantization and unitary representations*, Lecture Notes in Math., vol. 170, Springer-Verlag, Berlin and New York, 1970.

7. ______, *The solution to a generalized Toda lattice and representation theory* (to appear).

8. J. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Mathematical Phys. **5** (1974), 121–130.

9. C. L. Siegel and J. Moser, *Lectures on celestial mechanics*, Springer, New York, 1971.

10. J. M. Souriau, *Structure des systèmes dynamiques*, Dunod, Paris, 1970.

11. S. Sternberg, *Celestial mechanics*, Benjamin, New York, 1969.

12. ______, *Symplectic homogeneous spaces*, Trans. Amer. Math. Soc. **212** (1975), 113–130.

13. W. Thirring, *Classical dynamical systems*, Springer-Verlag, New York, 1978.

14. E. T. Whittaker, *A treatise of the analytical dynamics of particles and rigid bodies*, Cambridge Univ. Press, London, 1904–1965.

15. E. P. Wigner, *The unreasonable effectiveness of mathematics in the natural sciences*, Comm. Pure. Appl. Math. **13** (1960).

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