over algebraic number fields. The author chose **Q**. This has its pros and it has its cons. On the positive side, it enables the approach to be more elementary, the proofs are more concrete, there is no need to use results from class field theory which is difficult enough to understand let alone to develop, and it makes the material more accessible to mathematicians in other areas–group theory, combinatorics, topology, differential geometry–who in the past have found **Z** and **Q** good enough for their purposes. On the negative side it must be said that these same mathematicians are beginning to find **Z** and **Q** too specialized, that it is not as simple as the author suggests to extend things from **Q** to algebraic number fields, and that the reader who has mastered the subject over **Q** will be faced with a psychological barrier in having to go over it all again over an algebraic number field. My advice to the novice who intends to work in quadratic forms is, in fact, to start out over number fields.

So much for overall philosophy. Some other points should also be mentioned. Cassels emphasizes the effectiveness of the results whenever he can. This is a welcome feature of the book although, on one occasion, I found his explanation inadequate and unconvincing. Next, at the very end of the book he shows how the use of Dirichlet's theorem can be replaced by some elementary, but nontrivial, theory. He also shows that the folklore on the equivalence between the geometric and the form approach to spinor genera is true, a service to the expert, but incomplete and confusing to others. The author's development of Minkowski reduction and composition theory is clearly done and to be recommeded. My overall disappointments include a certain vagueness that is all too often covered by a wave of the hand, and an incompleteness that leaves you with the feeling that you have not been brought to the frontiers of research. Whether or not the decision to work over **Q** is a disappointment will depend on what you intend to use the book for.

The audience for *Rational quadratic forms* will be those mathematicians who wish to apply the arithmetic theory of quadratic forms and either want to learn the subject or have a good reference source for theorems over **Z**; students who wish to work in the theory; and specialists who are interested in seeing the subject from a somewhat different perspective. The ultimate questions are whether to buy the book; and, having bought it, whether to read it; and, in reading it, whether one will enjoy it. My answer to the first of these questions is yes; to the second, yes if you are just interested in **Z** or if you are looking for a different perspective; my answer to the third question is that I did.

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Compact right topological semigroups and generalizations of almost periodicity, by J. F. Berglund, H. D. Junghenn and P. Milnes, Lecture Notes in Math., vol. 663, Springer-Verlag, Berlin-Heidelberg-New York, 1978, x + 243 pp., $12.00.

This monograph in lecture-notes' clothing (hereafter referred to as BJM) has something in it for everyone: Semigroups S and the backchat between semigroup structure and topological structure; the space $B(S)$ of bounded complex functions on S, right and left translations, semigroup actions, invariant and introverted subspaces and subalgebras F of $B(S)$; invariant means on F and amenability; compactifications of S related to such F; categorical rephrasing of much of the preceding, with the adjoint functor theorem as an alternative tool for constructing compactifications; for climax the most rococo "GRAND DESIGN" extant; and at the end unusually helpful tables and indices.

Almost periodicity. The classical summation of the topic of almost periodic functions is the monograph of H. Bohr [1932] in which important forerunners of many of the topics of BJM are treated, of course from the viewpoint of a classical analyst: (1) There is a group, in this case the additive group R of real numbers. (2) The group R has its familiar topology, so the normed space $C(R)$ of bounded continuous functions is present. (3) There is a special family P of functions, in this case the functions, $\exp(i\lambda x)$, whose value in approximating other functions is widely recognized. (4) A definition is given for a larger class AP of functions–the almost periodic functions–here defined in terms of the properties of the set of almost periods of f among the translates of f. (5) It is shown that AP is a norm-closed, translation-invariant subalgebra of $C(R)$. (6) A mean value is defined on AP; that is, a linear functional μ on AP such that for each f in AP $\mu(f)$ is in the closed convex hull of the set of values of f. (7) The connection with periodicity is completed by proving that each f in AP is the uniform limit of trigonometric polynomials $s(x) = \Sigma_\lambda c(\lambda)\exp(i\lambda x)$, and that the values of λ needed can all be chosen from those where the Fourier coefficient $a_\lambda = \mu_x(f(x)\exp(i\lambda x))$ is not 0. (8) Parseval's formula and uniqueness are proved; $\mu(|f|^2) = \Sigma_\lambda |a_\lambda|^2$, and each AP function f is uniquely determined by its Fourier coefficients a_λ.

The big omission from Bohr's book is the characterization given by Bochner [1926] of an almost periodic function f as an element of $C(R)$ whose set of translations $\{f(st|t \in R\}$ is a conditionally compact subset of $C(R)$. Bochner [1933] also extended the compactness definition to apply to functions defined in R with values in a complete normed linear space.

von Neumann [1934] picked up Bochner's [1926] definition of AP and applied it to not necessarily topologized groups G. von Neumann gave a new method of calculating the mean value on AP and showed that the Peter-Weyl theory, for expansion of continuous functions on compact groups in terms of coefficients of finite-dimensional unitary representations of G, became the theory of expansions of almost periodic functions on a general topological group G. von Neumann then used the theorem of van der Waerden [1933] (that each bounded finite-dimensional representation of each semisimple Lie group is continuous) to show that each AP function on a semisimple Lie group is continuous.

Since AP functions on general topological G are like continuous functions on compact G, the natural next step was to look for a compact group G^- into which G could be plunged carrying its AP functions onto the continuous functions on G^-. van Kampen [1936] did a special case, and Tannaka [1937] produced such a G^- as the group dual to the semigroup of all finite-dimensional irreducible representations of G. Of course, when there are not enough almost periodic functions to separate points of G, the homomorphism carrying G into G^- is not an isomorphism.

Two side issues. Bochner and von Neumann [1936] showed that the theory of almost periodic functions could be extended to functions with values in a complete linear topological space L; in particular, they showed that if G is a metrisable locally compact group, if H is the space of Haar measurable functions of summable square, if $\mathfrak{L}$ is the algebra of all bounded linear operators from H into H with its weak operator topology, and if for each g in G, L_g is the unitary operator of left translation by g in H, then $g \to L_g$ is an almost periodic function from G into $\mathfrak{L}$ if and only if G is compact.

Another topic off the direct track to BJM is the construction of other types of almost periodic functions on R (defined by closing the set of periodic functions in R in some norm other than that of $C(X)$) due to Stepanov, Weyl, and Besicovich; see Maak [1967, p. 229], for appropriate definitions and references.

The monograph of Maak (original edition [1950], corrected printing [1967]) gives a combinatorial definition of almost periodic function, which like Bochner's, is independent of the topology on G.

Weak almost periodicity. Just before Maak's book appeared Eberlein [1949] gave a new twist to almost periodicity with the invention of weak almost periodic functions. He used his (then recent) proof that weak compactness of a set E in a Banach space B is equivalent to weak sequential compactness of E to show that continuous complex-valued functions vanishing at infinity are WAP, and that for each reflexive B, bounded representation $g \to U_g$ of G by linear operators on B, each β in B^*, and each b in B, the function $g \to (\beta(U_g b))$ is WAP.

Amenability. A different track into this topic is the study of left invariant means on (subspaces of) the space $B(S)$ of all bounded complex functions on S. von Neumann [1929], accounting for earlier results of Hausdorff and Banach, showed (but in the language of finitely-additive measures rather than of means) that the presence of an invariant mean on $B(G)$ has strong effects on the group structure of G; for example, G can not contain a free subgroup on two generators.

For clarity let us review some definitions; see for example, the reviewer's papers [1957], [1969] or Hewitt and Ross [1963, §17]. Let F be a linear subspace or C^*-subalgebra of $B(S)$ which includes the constant functions. An element μ of F^* is a mean on F if for each f in F, $\mu(f)$ is in the closed convex hull of the set of values of f. If S is also a semigroup, then left translation L_s is defined on $B(S)$ by: for each f in $B(S)$, $[L_s f](t) = f(st)$ for all t in S. F is left-invariant if for all s in S, $L_s(F) \subseteq F$. If F is left-invariant, a mean μ in F^* is a left-invariant mean on F if for each f in F, $\mu(L_s f) = \mu(f)$ for all s in S. F is called left amenable if there is at least one one left-invariant mean on F; a topologized semigroup S is called left amenable if there is at least one left-invariant mean on $C(S)$, the space of continuous functions in $B(S)$.

The set $M(F)$ of means on F is a weak* compact convex-set, and the set

LIM(F) of left-invariant means is either empty or a weak* closed convex subset of $M(F)$. Mitchell [1966] (for the case $F = B(S)$) asked when LIM(F) intersects $MM(F)$, the set of multiplicative means on F; F is called extremely left amenable when this occurs. Granirer [1965], [1967] and his students took up the study of such semigroups. In the same period locally compact groups were studied until most of what was known for discrete groups was available for locally compact groups. This specific topic was summarized in Greenleaf [1969], and also appears as part of the reviewer's summary [1969] of the state of amenability in semigroups.

Another topic appeared in the reviewer's paper [1961] showing a relation between fixed points and left-amenability. Mitchell [1966] and Granirer [1965] showed how this generalized to extreme left amenability.

Compactification. An important idea for BJM is that of compactification. The pattern chosen here uses conjugate spaces rather than operator algebras, and can be referred back to M. H. Stone [1937]. If S is a completely regular space and if $C(S)$ is the Banach space of all bounded continuous complex functions on S, let $C(S)^*$ be the conjugate space of $C(S)$. Then the unit ball of $C(S)^*$ is weak* compact and the evaluation map e: $S \to (C(S)^*, w^*)$, defined for each s in S by $e_s(f) = f(s)$ for all f in $C(S)$, carries S homeomorphically into the set $M(C(S))$, which is the positive face of the unit ball in $C(S)^*$. The weak* closure of $e(S)$ is the maximal compactification Ш(S) of S. Ш(S), usually called the Stone-Čech compactification of S, is defined here as in Stone's paper; Čech used an entirely different method to get the maximal compactification. Ш(S) can be characterized geometrically as the set of extreme points of the positive face of the unit ball in $C(S)^*$ or algebraically as the set of all multiplicative means on the commutative C^* algebra $C(S)$. From the Banach algebra point of view, if F is a C^*-subalgebra of $C(S)$, the evaluation functionals e_s can be restricted to F and Stone's representation can be replaced by Gel'fand's to get a compact space with a continuous image of S dense in it. Different choices of F give different compactifications. When S is a semigroup as well as a topological space, the subspaces of interest are selected by a combination of topological and algebraic conditions.

Semigroups with topology. The last topic vital to BJM is that of compact semigroups. Naturally it would be nice to have the multiplication jointly continuous, but this is too restrictive.

A general compendium of semigroups by Clifford and Preston [1961] summarized the topic of discrete semigroups. By then much work had gone into topological semigroups, in particular those with jointly continuous multiplication. Hoffmann and Mostert [1966] summarize this period; while the structure of such semigroups is enormously worse than that of compact groups, the structure of the minimal two-sided ideal is remarkably good. Rosen [1956] showed how this gave good information about amenability.

The need to relax the requirement of joint continuity comes from various directions. The one-point compactification of R has only separately continuous multiplication. de Leeuw and Glicksberg [1961] studied operator semigroups and pointed out that the weak or strong operator topology in the algebra $\mathfrak{L}(X)$ of all bounded linear operators from a Banach space X into itself has only separately continuous multiplication. Arens [1951] defined from the multiplication in a given Banach algebra A a new multiplication in the second conjugate algebra A^{**} of A, which made A^{**} a Banach algebra but which in the weak* topology of A^{**} had continuous right multiplication by all elements of A^{**} but continuous left multiplication only by some elements of A^{**}. If this is applied with $A = l_1(S)$, then the set $M(S)$ of means on $B(S)$ becomes a semigroup under Arens multiplication. If S is left amenable, LIM becomes a convex, weak* compact right-zero semigroup, in fact, $\mu * \nu = \nu$ for all μ in $M(S)$ and ν in LIM. When S is the additive group of the integers, this demonstrates that even starting out with the simplest infinite abelian group leads to a noncommutative compact semigroup $M(S)$ in which continuity is available on only one side.

The first summary of the theory of semigroups with separately continuous multiplication, by Berglund and Hoffmann [1967], remarked that there was not yet a second fundamental theorem in the field. (The first dealt with the properties and structure of the minimal ideal.) Here already in addition to the structure questions, there are topics important to BJM: almost periodic functions and weak almost periodic functions, fixed point properties, and categorical methods. The authors use operator semigroups as their usual method of compactification.

BJM at last. By the time BJM begins it is clear that compactification may spoil joint continuity and inverses, so the proper kind of object to include in the study is a right topological semigroup S; precisely defined, S has a topology and an associative multiplication and, for each t in S, right multiplication ρ_t, defined by $\rho_t(s) = st$ for each s in S, is continuous from S into S.

For various subspaces and subalgebras F of $C(S)$ two kinds of compactification are studied: Taking weak* closure of the set of e_s in F^*; this is especially appropriate when F is a subalgebra of $C(S)$ and gives $MM(F)$, the set of multiplicative means on F. The other construction gives the compact affine semigroup $M(F)$, the set of means on F, as the weak* closure of the convex hull of the set of all e_s.

In the special case $S = R$, we have already met in $C(S)$, the Banach algebra of bounded continuous functions on S, a number of important subspaces. Let R_t $[L_t]$ be defined for each f in $B(S)$ by $R_t(f) = f \cdot \rho_t$ $[L_t(f) = f \cdot \lambda_t]$. Then we have already met $AP(S)$, the set of f in $C(S)$ for which the set of translates $\{R_t f | t \in S\}$ is conditionally compact in the norm topology of $C(S)$, and $WAP(S)$, the set of f where $\{R_t f | t \in S\}$ is weakly conditionally compact in $C(S)$. Another pair of useful spaces to study are the spaces of [weakly] left uniformly continuous functions on S, defined by $f \in LUC(S)$ $[WLUC(S)]$ if and only if the function $s \to L_s f$ is continuous from S into $C(S)$ with its norm [weak] topology. Even in semigroups with jointly continuous multiplication, $AP(S)$, as just defined, and $SAP(S)$, the closed linear hull of the set of coefficients of finite-dimensional continuous unitary representations, may differ. An example uses [0, 1] with its usual topology but with $xy = y$ for x, y in S, to get $AP(S) = C(S)$ while the only unitary representation of S is $U(S) \equiv 1$. A total of eleven possibly distinct proper subspaces of $C(S)$ are considered in BJM, each selected by some combination of algebraic and topological properties. Then a critical technical property (called left-introversion, see §10 of the reviewer's [1956] paper) must be proved for each of these spaces F in order that a standard construction of a multiplication in F^* shall yield a product with values in F^*.

BJM organizes all this material beautifully. After a preliminary chapter on flows, means, and semigroups of means, follows a chapter on structure of compact, right-topological semigroups and compact affine right-topological semigroups. Compact right-topological groups have other good properties; for example, $\Lambda = \{s|\lambda_s$ is continuous$\}$ is a closed subgroup of G if G is metrisable. Affine semigroups are also better than general semigroups.

In Chapter III the compactifications $M(F)$ and $MM(F)$ are defined for general left-introverted subspaces or subalgebras F of $C(S)$. Eleven particular subspaces of $C(S)$ are selected, and shown to be left-introverted. Conditions on S are discussed under which some of these spaces coincide with others; for example, when G is a compact topological group, all eleven spaces equal $C(S)$. Conditions on $S \subseteq T$ are given that each element f of a subspace of one type for S shall be extendable to a function of the same type for T.

The old connection between ergodic theory and invariant means appears again here, where, under appropriate hypotheses on the subalgebra F (conditions satisfied, for example, by the space studied by de Leeuw and Glicksberg [1961], $F = WAP(S)$), F can be written as a direct sum $F = F_0 + F_r$, where F_0 is the subset of all f in F for which 0 is in the pointwise closure of $\{R_s f | s \in S\}$. These are the "flight vectors" of K. Jacobs [1960, p. 22]; F_0 is a closed linear subspace of F if and only if S has a unique minimal left ideal. F_r is the set of f in F such that if g is in the pointwise closure of $\{R_s f | s \in S\}$, then f is the pointwise closure of $\{R_s g | s \in S\}$. These are the "reversible vectors" of Jacobs, loc. cit.

In the reviewer's 1961 note, topologically tidied in 1964, it was shown that a discrete topological semigroup S is left-amenable if and only if each homomorphism h: $s \to h_s$, which represents each element s of S by an affine continuous mapping h_s of a compact convex subset K of some locally convex space L, has a common fixed point; that is, there is p in K such that $h_s(p) = p$ for all s in S.

One cannot blithely require continuity of the function $s \to h_s$ in order to adapt this to the topological case. To be more precise, call h slightly continuous if there is a point k in K for which $s \to h_s k$ is continuous from S into K. Then $C(S)$ is left amenable if and only if every slightly continuous representation has a fixed point. Mitchell [1966] adapted this to extremely amenable semigroups by dropping "affine" from the conditions above. Mitchell [1970] characterised the kind of F which has a multiplicative left-invariant mean in terms of the kind of representation h of S which must have a fixed point. Chapter IV of BJM gathers these results; for example, from Mitchell [1970], Lau [1973] $WLUC(S)$ [$AP(S)$] is left amenable if and only if every separately continuous affine representation $s \to h_s$ [with the family of h_s equicontinuous] has a fixed point.

Chapter V is a collection of examples which are used throughout the book to distinguish spaces F with different definitions. Appendix A defines an alternative approach to the selection and construction of compactifications by means of the adjoint functor theorem. Since category theory (beyond first and second) is a state of mind which this reviewer is too old to believe in, one can only say that it works and gives a new view of the same compactifications as before. A pattern midway through this appendix erects a tower of all the categories of semigroup modules and semigroups that anyone might ever want, faces this tower with the tower of subcategories of their compact objects, and braces these categories together by means of the adjoints of the inclusion functors to get THE GRAND DESIGN, whose details are clarified in the rest of the appendix.

To help the reader find important properties of specific compactifications the book closes with a table listing interesting facts about each space *F*. Indices and bibliography are complete and helpful. In short, the authors have written a scholarly monograph on this topic rather than an ephemeral set of notes with only some timeliness for excuse.

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Von Neumann regular rings, by K. R. Goodearl, Monographs and Studies in Mathematics, No. 4, Pitman, London-San Francisco-Melbourne, 1979, xvii + 369 pp., $46.00.

The conception of von Neumann regular rings occurred in 1936 when John von Neumann defined a *regular ring* as a ring R with the property that for each $a \in R$ there exists $b \in R$ such that $a = aba$. In order to distinguish these rings from the regular Noetherian rings of commutative algebra, noncommutative ring theorists have added von Neumann's name as a modifier. There is, however, very little chance of confusing these two concepts since their only common objects of study would be fields. The standard example of a regular ring is the complete ring of linear transformations of a vector space over a division ring.

Motivated by the coordinatization of projective geometry which was being reworked at that time in terms of lattices, von Neumann introduced regular rings as an algebraic tool for studying certain lattices. The lattices von Neumann was interested in had arisen in joint work with F. J. Murray dealing with algebras of operators on a Hilbert space [10], which subsequently came to be known as *von Neumann algebras* or *W^*-algebras*. Although a W^*-algebra A turns out to be a regular ring only when A is finite-dimensional, a regular ring can be assigned to A by working with the set $P(A)$ of projections, a projection on A being a selfadjoint idempotent. For a finite W^*-algebra A, Murray and von Neumann used a regular ring R to "coordinatize" $P(A)$ in the sense that $P(A)$ turned out to be naturally isomorphic to the lattice of principal right ideals of R. (Finite means that $tt^* = 1$ whenever $t^*t = 1$, for $t \in A$.) Expanding on this idea [14], von Neumann invented regular rings so as to coordinatize complemented modular lattices, a lattice L being coordinatized by a regular ring R if it is isomorphic to the lattice of principal right ideals of R. As von Neumann showed, almost all complemented modular lattices could be coordinatized by a regular ring.

The roots of regular rings were firmly embedded in the theory of operator algebras and lattice theory. From the purely ring-theoretic viewpoint regular rings as a subject of investigation were largely ignored for a long period of time. In N. Jacobson's bible for ring theorists [5], regular rings are mentioned only briefly (p. 210). Yet there were intimations that regular rings might be worthy of study for their own sake, since they appeared in various contexts.