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Set theory, by Thomas Jech, Academic Press, New York, 1978, xii + 621 pp., $53.00.

"General set theory is pretty trivial stuff really" (Halmos; see [H, p. vi]). At least, with the hindsight afforded by Cantor, Zermelo, and others, it is pretty trivial to do the following. First, write down a list of axioms about sets and membership, enunciating some "obviously true" set-theoretic principles; the most popular list today is called ZFC (the Zermelo-Fraenkel axioms with the axiom of Choice). Next, explain how, from ZFC, one may derive all of conventional mathematics, including the general theory of transfinite cardinals and ordinals.

This "trivial" part of set theory is well covered in standard texts, such as [E] or [H]. Jech's book is an introduction to the "nontrivial" part.

Now, nontrivial set theory may be roughly divided into two general areas. The first area, *classical set theory*, is a direct outgrowth of Cantor's work. Cantor set down the basic properties of cardinal numbers. In particular, he showed that if κ is a cardinal number, then 2^κ, or $\exp(\kappa)$, is a cardinal strictly larger than κ (if A is a set of size κ, 2^κ is the cardinality of the family of all subsets of A). Now starting with a cardinal κ, we may form larger cardinals $\exp(\kappa)$, $\exp_2(\kappa) = \exp(\exp(\kappa))$, $\exp_3(\kappa) = \exp(\exp_2(\kappa))$, and in fact this may be continued through the transfinite to form $\exp_\alpha(\kappa)$ for every ordinal number α. These considerations naturally led to investigations on a number of different fronts. The earliest dealt with the obvious question of whether there are any cardinals betwen κ and 2^κ. The GCH (Generalized Continuum Hypothesis) is the statement that for all infinite κ, $2^\kappa = \kappa^+$ (κ^+ is the next cardinal larger than κ). The CH is the special case, $2^{\aleph_0} = \aleph_1$, where $\aleph_0$ is the smallest infinite cardinal, or the cardinality of the set of integers, and $\aleph_1 = (\aleph_0)^+$. There were extensive investigations in the 1920s and 30s of consequences of CH, or of its negation, without yielding any insight into whether CH was really true or false. Another front is *large cardinals*, or cardinals whose size transcends those which can be produced on the basis of the ZFC axioms alone. The smallest large cardinal is an inaccessible cardinal. If κ is inaccessible, then, among other things, $\kappa > \exp_\alpha(\aleph_0)$ for any finite or countable α, or for any α of size less than κ. Measurable cardinals, which are much larger, arose naturally from measure-theoretic considerations. A third front is *infinitary combinatorics*. Once one has a "transfinite arithmetic", it is natural to consider the analogs for infinite cardinals of various questions in finite combinatorics. For example, transfinite Ramsey theory has been extensively developed by Erdös and others. A fourth front, *descriptive set theory*, grew out of a detailed study of Borel and analytic sets of real numbers, and, after Kleene's work in the 50s, was seen to be closely related to recursion theory.

The second area is *independence proofs*. Here instead of trying to prove a statement, S, from ZFC, we try to show that S is *not provable* from ZFC; equivalently, that ZFC plus the negation of S is consistent (assuming always that ZFC is consistent). S is called *independent* of ZFC iff neither S nor its negation is provable from ZFC. Such results, involving as they do the consistency of formal axiomatic systems, always require some elementary use of mathematical logic for their rigorous explication. The first consistency result of mathematical significance was Gödel's proof, around 1938, that the GCH is consistent with ZFC. Much later, around 1963, Cohen showed that the negation of GCH (and in fact of CH), is also consistent with ZFC. It was perhaps Cohen's proof which had the greater influence on mathematics; Gödel's proof produced just one consistency result (or one model), whereas Cohen's method of *forcing*, as later expanded by Solovay and others, was seen to apply to produce a wide variety of consistency results (or, many different models), showing that a large number of famous open problems of set theory, beginning with CH, were in principle not decidable from ZFC. Set-theorists can now investigate the consequences of ZFC augmented by the various additional axioms which have been shown to be consistent with it. These results also affected many other branches of mathematics, such as general topology and measure theory, where the questions of classical set theory have some relevance.

The above outline of set theory also roughly describes the contents of Jech's book. His primary emphasis is on forcing and large cardinals, but there is a substantial discussion of descriptive set theory and infinitary combinatorics as well.

In forcing, he starts with the basics and goes through such topics as perfect set forcing, the consistency of Martin's axiom, and the independence of Kurepa's hypothesis. In large cardinals, he covers, among other things, measurable cardinals, $0^{\#}$, and iterated ultrapowers. He also relates large cardinals with forcing; for example, there is a detailed discussion of the consistency strength of the failure of GCH at a measurable cardinal. In descriptive set theory, he covers basic consequences of the Axiom of Determinateness, such as the measurability of $\aleph_1$ and $\aleph_2$. Finally, he covers many of the elementary topics in infinitary combinatorics, such as trees, stationary sets, the delta-system lemma, and the Erdös-Rado theorem.

The author's presentation is in general very well-organized and carefully worked out. The only serious error I could spot was in the construction of a model with no selective ultrafilters (p. 481), where the author makes the common mistake of confusing the measure-theoretic product of two measure algebras with the Boolean algebraic completion of the product order. The former is a measure algebra, while the latter is not. There is then the resultant error in discussing iterated forcing with measure algebras.

The author does not state explicitly what he intends as prerequisites for reading his book. Ideally, the reader should be familiar both with the elementary development of ZFC (as in [E] or [H]) and with some logic. Actually, the reader's knowledge of ZFC could be rather sketchy, since the review given in Chapter 1 is quite thorough. As for logic, the reader should know model theory through the completeness theorem and the downward Löwenheim-Skolem-Tarski theorem, and should be familiar with the relationship between objects in the metatheory and objects in the formal theory. There is a brief review of these matters on pp. 80–82, but this review might leave the nonlogician confused as to precisely what a model is, and what the distinction is between $\langle M, E\rangle \models \ulcorner\varphi\urcorner$ and $\langle M, E\rangle \models \varphi$; here, φ is a sentence in the language of set theory, $\ulcorner\varphi\urcorner$ is the corresponding constant in the formal theory, and there is an unfortunate typographical error in the key display (10.2) describing their relationship (the second φ should be a $\ulcorner\varphi\urcorner$).

Despite these minor criticisms, this is a very fine book. It collects an enormous amount of material on forcing and large cardinals which had previously been available only through scattered journal articles, or, in some cases, by private communication. The book will be extremely valuable used either as a reference or as an introduction to modern set theory.

REFERENCES

[E] H. B. Enderton, *Elements of set theory*, Academic Press, New York, 1977.

[H] P. R. Halmos, *Naive set theory*, The University Series in Undergraduate Mathematics, Van Nostrand, Princeton, N.J.-Toronto-London-New York, 1960.

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Errata to

Crystallographic groups of four-dimensional space, H. Brown, R. Bülow, J. Neubüser, H. Wondratschek and H. Zassenhaus, Volume 1, Number 5, September 1979, pp. 792–794.

On p. 793 it was implied that the Moors, in their decoration of the Alhambra, never used the symmetry groups p2 and pm. J. J. Burckhardt has pointed out that two of their patterns of intersecting circles are colored with five colors in such a way that one of them exhibits the symmetry p2, and the other pm. In Edith Müller's famous thesis, *Gruppentheoretische und Strukturanalytische Untersuchungen der Maurischen Ornamente aus der Alhambra in Granada* (Ruschlikon, 1944, 128 pp., 43 plates), these two patterns are numbered 19 and 20 on Tafel 9, between pp. 60 and 61.

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