9. R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloq. Publ., no. 19, Amer. Math. Soc., New York, 1934.

10. J. L. Walsh, *A generalisation of the Fourier cosine series*, Trans. Amer. Math. Soc. **22** (1921), 230–239.

11. K. Yao and J. B. Thomas, *On some stability and interpolatory properties of nonuniform sampling expansions*, IEEE Trans. Circuit Theory CT-**14** (1967), 404–408.

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Polynomial identities in ring theory, by Louis Halle Rowen, Academic Press, New York, 1980, xx + 365 pp., $39.50.

Rings satisfying a polynomial identity (PI) occupy an important place in modern noncommutative ring theory. Louis Rowen's book is an up to date, thorough presentation of PI-theory and related topics.

PI-theory was launched by a paper of Kaplansky in 1948 and "born again" with the discovery, by Formanek in 1971, of central polynomials. For convenience in this review we shall assume that our rings are algebras over fields. Thus, if A is an algebra over the field F, we say that A is a PI-ring if there exists a nonzero polynomial $p(x_1, \ldots, x_t)$ in noncommuting indeterminates with coefficients in F such that $p(a_1, \ldots, a_t) = 0$ for all possible substitutions of elements $a_1, \ldots, a_t$ in A. We say that p is an identity for A and that A is a PI-ring of degree d if d is the least degree of a polynomial which is an identity for A.

Commutative rings, obviously, satisfy $x_1x_2 - x_2x_1$. Nilpotent rings satisfy x^N where N is the index of nilpotence of the ring. Subalgebras and factor algebras of PI-rings are also PI-rings.

Before proceeding to the most important class of PI-rings, we pause for a definition. The nth standard polynomial S_n is $\Sigma_{\sigma \in \mathbb{S}_n} \operatorname{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)}$ where the sum runs over the symmetric group, $\mathbb{S}_n$, on n letters and the x's are noncommuting variables. A celebrated theorem of Amitsur and Levitzki asserts that S_{2n} is an identity for the $n \times n$ matrices over a commutative ring. It is not difficult to see that $2n$ is the least degree of a polynomial identity for $n \times n$ matrices. Therefore, factor algebras of subalgebras of matrices are PI-rings–though not all PI-rings arise this way.

An important theme in the theory of PI-rings is the study of the "closeness" of classes of PI-rings to matrices over commutative rings and finding "tight" connections between PI-rings and their centers. Kaplansky proved that a primitive PI-ring is simple and finite dimensional over its center which is a field. Amitsur later showed that PI-rings with no nonzero nilpotent ideals are embeddable in matrices over commutative rings. To put this last result in perspective we note an observation of P. M. Cohn: the exterior algebra on an infinite-dimensional vector space over a field of characteristic 0 satisfies no standard polynomial and, hence, cannot be a subalgebra of matrices over a commutative ring.

We turn now to the second "birth" of the subject. It is straightforward that $(x_1x_2 - x_2x_1)^2$ is not a polynomial identity for 2×2 matrices over a commutative ring, but all values of this polynomial lie in the center. Such "central polynomials" were unknown for larger matrices until Formanek (and Razmyslov, independently) in 1971 constructed them for all n. The existence of central polynomials implies that PI-rings with no nonzero nilpotent ideals have large centers. Additionally, central polynomials lead to better proofs and generalizations of two important theorems: Posner's theorem and Artin's theorem.

Posner's theorem, in its generalized form, says that a prime (a ring in which the product of nonzero ideals is nonzero) PI-ring has a ring of fractions which is a finite-dimensional central simple algebra. The "denominators" are the nonzero central elements and the center of the ring of fractions is the quotient field of the center. Artin's theorem gives a condition solely in terms of identities which forces a ring to be a so-called rank n^2 Azumaya algebra–a generalization of matrices over a commutative ring.

The supreme achievement of PI-theory is Amitsur's construction of a finite-dimensional division algebra which is not a crossed product, that is, for which there is no maximal subfield which is a Galois extension of the center. PI methods have been used since then to solve other important problems in division algebras. Rowen's book gives a good account of these results.

Finally, in this brief tour of PI-theory, let me mention the work of Regev and others on the identities of matrix rings which has led to an interesting dialogue with combinatorics. PI-theory is shedding new light on certain aspects of the representation theory of the symmetric group–and, conversely, representation theory yields precise results on the nature of certain identities.

Rowen's book treats all of the above topics and *much* more (rational identities, generalized identities, nonassociative questions, etc.), usually *ab initio*. Herein lies a problem: the author must do everything *his* way and with *his* names ("characteristic closure" for trace ring, "reduced length" for reduced rank, etc.) and *his* notation. Some well-known results become hard to decipher. The expert will cope and find the book an indispensable reference; the student will appreciate the very few prerequisites; it is the casual, browsing algebraist who will be frustrated by unfamiliar terminology and symbols.

The author's style is reasonably chatty, with an occasional Small gesture towards humor ("a very Small example", for instance). There are many exercises which extend the text and a large number of examples and counter-examples. *Polynomial identities in ring theory* is a very welcome addition to the literature and should have a place on the bookshelf of everyone who has an interest in ring theory.

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