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Function theory in the unit ball of $\mathbf{C}^n$, by Walter Rudin, Grundlehren der mathematischen Wissenschaften, Band 241, Springer-Verlag, Berlin and New York, 1980, xiii + 436 pp., $39.00.

0. Introduction. It has often been remarked that the subject of real analysis had a fallow period during the 1930's and 1940's. Some have laid the blame, perhaps unjustly, at the feet of Hardy and Littlewood. It is said that they could see little interest in doing the theory of one variable with the additional clap-trap of multi-indices. So a lot of time and energy was instead expended in constructing exotic counterexamples and exploring remote corners of the theory of one variable. Immediate contradictions to what I have just said spring to mind: Zygmund, Marcinkiewicz, Saks, Wiener, Bochner, and many others did powerful and significant work during this period. But it is safe to say that while algebra, algebraic topology, and algebraic geometry were developing very rapidly from 1930–1950, real analysis was not making the (what by now seems) obvious move into the theory of several variables.

In retrospect, it is easy to understand how this recession in real analysis came about. The differences between real analysis of one and several variables–subellipticity, propagation of singularities, the existence of singular integrals, the failure of the multiplier problem for the ball, the connections between covering theorems and the boundedness of integral operators, restriction theorems for the Fourier transform, the theory of currents and the solution of the general Plateau problem, etc.–all lie very deep. It is amazing that *anyone* discovered these phenomena, much less mathematicians who believed that there were no phenomena to discover. Hardy and Littlewood could not have conceived what is now painfully clear: that $\mathbf{R}^1$ is the exceptional dimension, $\mathbf{R}^3$ is typical, and $\mathbf{R}^2$ is some intermediate bastardization.

Complex analysis has enjoyed rather a different history. Hartogs discovered quite early (1906) the phenomenon of domains of holomorphy. Recall that a holomorphic function of several complex variables is one which is holomorphic (in the one-variable sense) in each variable separately. Then Hartogs's result is

THEOREM 1 (HARTOGS). *Let*

$$\mathbf{C}^2 \supseteq \Omega \equiv \{|z_1| < 1, \quad |z_2| < 1\} \setminus \{|z_1| \leqslant 1/2, \quad |z_2| \leqslant 1/2\}.$$

Let $f\colon \Omega \to \mathbf{C}$ *be holomorphic. Then there is a holomorphic*

$$\hat{f}\colon \{|z_1| < 1, \quad |z_2| < 1\} \to \mathbf{C}$$

such that $\hat{f}|_\Omega = f$.

PROOF. For each fixed z_1, $|z_1| < 1$, write f as a Laurent series in z_2,

$$f(z_1, z_2) = \sum_{n=-\infty}^{\infty} a_n(z_1) z_2^n,$$

where

$$a_n(z_1) = \frac{1}{2\pi i} \int_{|\zeta|=3/4} \frac{f(z_1, \zeta)}{\zeta^{n+1}}\, d\zeta.$$

Notice that a_n is a holomorphic function of z_1. For $1/2 < |z_1| < 1$, $f(z_1, \cdot)$ is holomorphic so $a_n(z_1) = 0$ for $n < 0$. By analytic continuation, $a_n \equiv 0$ for $n < 0$. So $\hat{f}(z_1, z_2) \equiv \sum_{n=0}^{\infty} a_n(z_1) z_2^n$ defines the desired extension. □

This contrasts sharply with the situation in one variable.

THEOREM 2. *Let $\Omega \subsetneq \mathbf{C}^1$ be any open set. There is a holomorphic f: $\Omega \to \mathbf{C}$ which cannot be analytically continued to any larger open set.*

PROOF. Let $\{z_j\}_{j=1}^{\infty} \subseteq \Omega$ accumulate at every point of $\partial\Omega$ but at no point of Ω. By Mittag-Leffler's theorem, there is a holomorphic f: $\Omega \to \mathbf{C}$ which satisfies $f^{-1}(\{0\}) = \{z_j\}$. If f continued holomorphically past any $z \in \partial\Omega$, then $f^{-1}(\{0\})$ would have z as an interior accumulation point so $f \equiv 0$. □

Finally notice that if $\Omega \subseteq \mathbf{C}$ is a bounded open set and f is as in Theorem 2 then F: $\Omega \times \Omega \to \mathbf{C}$ given by $F(z_1, z_2) = f(z_1)f(z_2)$ is holomorphic in each variable separately but will not analytically continue past any point of $\partial(\Omega \times \Omega)$. So the problem arises of characterizing those domains in $\mathbf{C}^n$ which have the continuation property of Theorem 1.

While other interesting phenomena were discovered, such as Poincaré's theorem that the ball $\{|z_1|^2 + |z_2|^2 < 1\}$ and the polydisc $\{|z_1| < 1, |z_2| < 1\}$ are biholomorphically inequivalent, it is safe to say that Hartog's phenomenon had *the* major influence over the development of the theory of several complex variables in the first half of this century. To explain what was done, we introduce some auxiliary concepts. In what follows, $\Omega \subseteq \mathbf{C}^n$ is a connected open set and $\mathcal{F} = \mathcal{F}(\Omega)$ the set of holomorphic functions on Ω.

DEFINITION 1. If $K \Subset \Omega$, define

$$\hat{K} = \left\{ \hat{z} \in \Omega: |f(\hat{z})| \leqslant \sup_{z \in K} |f(z)|, \quad \text{all } f \in \mathcal{F} \right\}.$$

Call Ω *holomorphically convex* if $K \Subset \Omega$ implies $\hat{K} \Subset \Omega$.

DEFINITION 2. Assume that $\Omega \subseteq \mathbf{C}^n$ has C^2 boundary. Call a C^2 function ρ: $\mathbf{C}^n \to \mathbf{R}$ a *defining function* for Ω if $\Omega = \{z \in \mathbf{C}^n: \rho(z) < 0\}$ and $\nabla\rho \neq 0$ on $\partial\Omega$.

DEFINITION 3. Let Ω, ρ be as in Definition 2. Let $z \in \partial\Omega$ and $w \in \mathbf{C}^n$. Call w *tangential* to $\partial\Omega$ at z if $\sum_{j=1}^{n} (\partial\rho/\partial z_j)(z) w_j = 0$.

DEFINITION 4. Let Ω, ρ be as in Definition 2. Call Ω *Levi pseudoconvex* if

$$(*) \qquad \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) w_j \bar{w}_k \geqslant 0$$

for all $z \in \partial\Omega$ and all w which are tangential at z.

Definitions 2 and 3 are self-explanatory. To understand Definition 1, consider its real variable analogue. Let $\Omega \subseteq \mathbf{R}^N$ and $\mathcal{G}$ be the family of real-valued linear functions (on $\mathbf{R}^N$). Then Ω is $\mathcal{G}$-convex if and only if Ω is convex in the classical sense. For Definition 4, the real variable analogue is as follows. If $\Omega \Subset \mathbf{R}^N$ has C^2 boundary and ρ is a defining function then

$\alpha = (\alpha_1, \ldots, \alpha_N)$ is called *tangential* at $x \in \partial\Omega$ if $\Sigma(\partial\rho/\partial x_j)(x)\alpha_j = 0$. Then Ω is convex if and only if $\Sigma(\partial^2\rho/\partial x_j\partial x_k)(x)\alpha_j\alpha_k \geqslant 0$ for all $x \in \partial\Omega$ and α tangential at x. The expression on the left side of (*) (called the *Levi form*) is the holomorphically invariant analogue of this real Hessian.

Now let us return to our subject proper. It is a nice insight, but not exceedingly difficult, to prove that Ω is holomorphically convex if and only if there is a holomorphic f on Ω which cannot be continued to any larger open set. Call such an Ω a *domain of holomorphy*. It is also not too hard to see that a domain of holomorphy with C^2 boundary must be Levi pseudoconvex. Here is an intuitive proof of this last assertion. If Ω is not Levi pseudoconvex then some $z \in \partial\Omega$ has Levi form with a negative eigenvalue. Thus $\partial\Omega$ is not convex in some complex direction. As a result there is a holomorphically imbedded disc $\mathfrak{d}$ as shown:

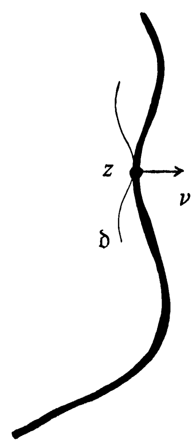

FIGURE 1

If ν is the unit outward normal at P, then $K = \bigcup_{j=1}^{\infty} \partial(\mathfrak{d} - \nu/j)$ is relatively compact in Ω while $\hat{K} \supset \bigcup_{j=1}^{\infty}(\mathfrak{d} - \nu/j)$ is not.

The *Levi problem* consists in showing that Levi pseudoconvex domains are domains of holomorphy. This problem arose from E. E. Levi's work in 1910 and was resolved in 1954 by Oka, Bremermann, and Norguet. Given the milieu which we described at the beginning of this review, it is not surprising that the principal tools which were developed to study the Levi problem were algebraic: the Cousin problems and sheaf cohomology (the latter is a far-reaching generalization of the former). While classical analysis played *some* rôle in the development of the subject of several complex variables in the first half of the century (c.f. for instance the book of Bochner and Martin [1]), it had little influence over the Levi problem and hence was not part of the mainstream. As a result, most courses on the subject of several complex variables focus on the theory of sheaves leading up to Cartan's Theorems A and B. The book of Gunning and Rossi [5], commonly hailed as the first modern treatise on several complex variables, represents this algebraic viewpoint. It accurately reports the significant developments up to the time of its writing, and has strongly influenced the way most people have learned the subject since. In particular, classically-oriented analysts have often found several complex variables uninviting.

New influences came into play around the time that [5] was being written. To explain what they were, we shall pass over the theories of complex manifolds, Stein spaces, analytic vector bundles, complex differential geometry, and a number of other significant branches of the theory, and focus on two important developments.

1. The Cauchy-Riemann equations. If $z \in \mathbf{C}^n$, write $z = (z_1, \ldots, z_n)$ and $z_j = x_j + iy_j$. Define the differential operators

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Let $\Omega \subseteq \mathbf{C}^n$ be open and let $u \in C^1(\Omega)$. Then u is holomorphic in the variable z_j if and only if u satisfies the Cauchy-Riemann equations in the variable z_j which is true if and only if $\partial u/\partial \bar{z}_j \equiv 0$ on Ω.

Let $dz_j = dx_j + idy_j$, $d\bar{z}_j = dx_j - idy_j$, $j = 1, \ldots, n$. Define $\bar{\partial} u = \sum_{j=1}^n (\partial u/\partial \bar{z}_j) d\bar{z}_j$. Then u is holomorphic on Ω if and only if $\bar{\partial} u \equiv 0$ on Ω.

Since the solutions on Ω of $\bar{\partial} u = 0$ are important it is plausible that the following more general problem is also of interest. If $1 \leqslant q \leqslant n$ and $I = (i_1, \ldots, i_q)$ is a multi-index, let $d\bar{z}_I = d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_q}$. A $(0, q)$ form on Ω is an expression $\beta = \sum_I \beta_I(z) d\bar{z}_I$ where each $\beta_I \in C^\infty(\Omega)$ and the sum is over I's of length q. Of course $(0, 0)$ forms are just functions. Define $\bar{\partial}\beta = \sum \bar{\partial}\beta_I \wedge d\bar{z}_I$. Then the problem is, given a $(0, q)$ form α, to find a $(0, q - 1)$ form u with $\bar{\partial} u = \alpha$. Since $\bar{\partial}^2 = 0$, it is necessary that $\bar{\partial}\alpha = 0$. It turns out that this is sufficient for all $1 \leqslant q \leqslant n$ precisely when Ω is a domain of holomorphy. Thus the (historically) fundamental problem of the subject is rendered as a problem in partial differential equations. By certain classical reduction procedures, it is sufficient for many purposes to study the system $\bar{\partial} u = \alpha$ on strongly pseudoconvex domains (here a C^2 bounded domain is *strongly pseudoconvex* if the Levi form is positive definite—not semidefinite—at each point of $\partial\Omega$). After important developmental work by Spencer and Morrey, Kohn succeeded in 1964 in proving both existence and regularity theorems (in the Sobolev norm) for the $\bar{\partial}$ equation on a strongly pseudoconvex domain. Hörmander developed a different theory which applies to all pseudoconvex domains.

Here is a typical application of the technique of the $\bar{\partial}$ equation.

THEOREM 3. *Let $\Omega \subset \mathbf{C}^n$, $n > 1$. Suppose that for every $\alpha = \sum \alpha_j(z) d\bar{z}_j$ on Ω, α_j smooth with $\bar{\partial}\alpha = 0$, there is a smooth u on Ω with $\bar{\partial} u = \alpha$. Let f be a holomorphic function on $\omega \equiv \Omega \cap \{z_n = 0\}$. Then there is a holomorphic F on Ω with $F|_\omega = f$.*

PROOF. Let $\pi(z) = (z_1, \ldots, z_{n-1}, 0)$, $z \in \mathbf{C}^n$. Let $B = \{z \in \Omega: \pi(z) \notin \Omega\}$. Then B and ω are disjoint relatively-closed subsets of Ω. Let $\varphi \in C^\infty(\Omega)$

satisfy $\varphi \equiv 1$ on a relative neighborhood of ω and $\varphi \equiv 0$ on a relative neighborhood of B. We seek an F of the form

$$F(z) = \varphi(z)f(\pi(z)) - z_n \cdot u(z),$$

where u must be selected to make F holomorphic. This leads to the condition

$$0 = \bar{\partial} F = (\bar{\partial}\varphi) \cdot f(\pi(z)) - z_n \cdot \bar{\partial} u(z)$$

or

$$\bar{\partial} u(z) = f(\pi(z)) \cdot \bar{\partial}\varphi(z)/z_n \equiv \alpha(z).$$

Then α is smooth since $\bar{\partial}\varphi = 0$ on a neighborhood of ω. Also $\bar{\partial}\alpha = 0$. By hypothesis, there is a u solving this equation and we are done. $\square$

2. Integral representations. The Cauchy integral formula of one complex variable is powerful because it reproduces holomorphic functions. What is perhaps more significant, however, is that

$$F(z) = \frac{1}{2\pi i} \int \frac{d\mu(\zeta)}{\zeta - z}$$

is *holomorphic* off supp μ for any finite Borel measure μ. Via an elementary argument with Stokes's theorem, this crucial fact yields an integral formula for solutions of the $\bar{\partial}$ equation.

THEOREM 4 (SEE [6]). *Let $\alpha \in C_c^1(\mathbf{C}^n)$. Then the function*

$$u(z) = -\frac{1}{2\pi i} \int \frac{\alpha(\zeta)}{\zeta - z}\, d\bar{\zeta} \wedge d\zeta$$

satisfies $\bar{\partial} u = \alpha d\bar{z}$.

Since the compatibility condition $\bar{\partial}(\alpha(z)d\bar{z}) = 0$ is vacuous in $\mathbf{C}^1$, it is easy to extend the formula in Theorem 4 to $\alpha \in L^\infty$ with compact support, say, and to prove optimal estimates.

Matters are considerably less simple in several variables. While the formula of Bochner and Martinelli (see [4]) is valid on any C^2 domain in any $\mathbf{C}^n$, it does not have a holomorphic kernel (except in the case $n = 1$ when it reduces to the classical Cauchy formula). It was only in 1970 that Henkin and Ramirez succeeded in constructing an integral representation formula *with holomorphic kernel* for holomorphic functions on a smooth strongly pseudoconvex domain. As a result of these ideas, Henkin, Grauert-Lieb, Kerzman, and Øvrelid were able to obtain integral formulae for solutions to the $\bar{\partial}$ equation on strongly pseudoconvex domains. Very precise estimates for solutions to the equation could then be computed, and many function-theoretic problems attacked.

3. Classical analysis and the theory of several complex variables. Walter Rudin begins his book *Function theory in the unit ball of* $\mathbf{C}^n$ with the assertion: "Around 1970, an abrupt change occurred in the study of holomorphic functions of several complex variables. Sheaves vanished into the background, and attention was focused on integral formulas and on the 'hard analysis' problems that could be attacked with them . . . ."

This is a forgiveable hyperbole. The algebraists have hardly faded from the picture. On the other hand, consider the following statistics: in 1958 the *Math. Reviews* had no classification for "Several Complex Variables". In 1959 there were 85 items under that classification. In 1979 there were 747 items—nearly a nine-fold increase.

Since this jump in output can only in small part be attributed to an increase in the number of mathematicians as a whole or an increase in the productivity of existing mathematicians, the explanation for it lies elsewhere. Our thesis (and Rudin's) is that the theory of the $\bar{\partial}$ equation and the construction of integral formulae changed the face of the subject. This is not to say that everything done since 1970 uses these two ideas—far from it. Much of the best work on plurisubharmonic exhaustion functions, peaking functions, and the geometric structure of pseudoconvex domains requires little more than the definitions and an extraordinary amount of insight. On the other hand, the study of boundary behavior of biholomorphic maps, the theory of approximation of functions, the study of function algebras on domains in $\mathbf{C}^n$, the study of zero sets of functions in the Hardy and Nevanlinna classes, the boundary analysis of holomorphic functions—i.e., the *matière* of the classical analyst—have all received an enormous impetus from the developments described in §§1 and 2.

So in the late '60's and early '70's harmonic analysts, function algebraists and complex function theorists were given a theoretical framework in which they could relate to several complex variables. Hörmander's classic book [**6**] constitutes a good introduction to the $\bar{\partial}$ equation and the point of view it engenders. But it does not, indeed for chronological reasons it *could not*, describe most of the function-theoretic developments to which we have alluded above. There is a need for a book which does so, and Rudin's book addresses this need.

4. Rudin's book. It is not surprising that classical analysts, when approaching several complex variables, would try to use the ideas they already know to understand the new subject. This point of view has been extraordinarily successful on the ball and the polydisc for two reasons: (i) they both are closely related, especially via Schwarz's lemma, to the disc, and (ii) they both have a transitive group of biholomorphic self maps. A remarkable result of Bun Wong and Rosay shows that matters are rather different for arbitrary open sets in $\mathbf{C}^n$.

THEOREM 5. *Let $\Omega \Subset \mathbf{C}^n$ be an open set with C^2 boundary. If Ω has a transitive automorphism group then Ω is biholomorphic to the ball.*

The importance of this result lies in the fact that "most" domains are not biholomorphic to the ball. More precisely, let us topologize the collection of all C^∞ strictly pseudoconvex domains by using the C^∞ topology on defining functions. Call a domain *rigid* if its only biholomorphic self map is the identity. The following is a consequence of work of Burns-Shnider-Wells and Greene-Krantz.

THEOREM 6. *The collection of smoothly-bounded rigid strongly pseudoconvex domains is dense and open in the collection of all smoothly-bounded strictly pseudoconvex domains. In particular, the rigid strictly pseudoconvex real analytic domains are dense.*

So the study of the ball is the study of a very special domain indeed. In one variable, there is little loss of generality to do analysis on the disc because the disc is biholomorphic to any simply connected proper domain. But the ball is biholomorphic to virtually nothing else.

The advantage of studying the ball is that with essentially no specialized preparation one can immediately appreciate many differences between one and several complex variables. But *what is absolutely essential to understand* is that, for the most part, the differences one discovers on the ball are those that arise from the presence of several complex directions, *not* those that arise from the complex geometry of the boundary and the Levi form. (Still we must make a *caveat*: important work of Folland and Stein [3] and Fefferman [2] has shown that a strongly pseudoconvex boundary point has many of the significant properties of the boundary of the ball. Function theorists have not exploited this point of view as much as they might.)

Rudin's book does not consider the subject of several complex variables as it has traditionally (i.e., prior to 1964) been studied. It also does not consider the full force of what has taken place in the last fifteen years. But it purports to do neither. Rather, it presents those ideas, mostly function-algebraic in character, which can be developed fairly quickly and which demonstrate that the ball is not merely the disc with multi-indices. As Rudin points out in his preface, he *does* occasionally consider arbitrary domains in $\mathbf{C}^n$; however he does so *only* when the additional generality involves no additional work, i.e. only when he can avoid doing Levi geometry.

Rudin's point of view is particularly successful in his presentation of the Henkin-Skoda theorem which characterizes the zero sets of functions of the Nevanlinna class. This is a theorem whose full statement and proof (on strongly-pseudoconvex domains) involves such subtle methods as the theory of currents. On the ball, Rudin is able to give a fairly clean presentation and, by avoiding many technicalities, do his readers a great service. Similar success is achieved with some of the results on peak interpolation sets.

It is a truism for function theorists of several complex variables that the ball is a proving ground for theorems about strongly pseudoconvex domains. But in many instances the generalization of a theorem from the ball to the strongly pseudoconvex case is quite hard and involves new techniques (such as [3]). Rudin's decision to concentrate on the ball abrogates the question of how the boundary geometry influences the function theory on the interior. That it still reveals interesting new phenomena is a triumph both of technique and insight. But, by and large, the point of view Rudin has chosen is to see how one-variable questions undergo metamorphosis in $\mathbf{C}^n$ rather than to see what new questions might be posed.

The reader who approaches the book with the correct expectations will find much that is pleasing and informative. Many important ideas, such as the Bergman and Szegö integrals, integral formulae for the $\bar{\partial}$ equation, spherical harmonics, and tangential Cauchy-Riemann equations are neatly introduced and quickly applied. The student will find this feature, together with the very concrete nature of the theorems, most appealing. It is refreshing to see a book on this subject with coordinates, with computations, and with theorems that have brief statements with immediate impact. Let us give one illustration.

If $B \subseteq \mathbf{C}^n$ is the ball, $0 < p < \infty$, let $H^p(B) = \{f$ holomorphic on B: $\sup_{0<r<1} \int |f(r\zeta)|^p \, d\sigma(\zeta) < \infty\}$. Here σ is rotationally-invariant area measure on ∂B. Recall that when $n = 1$ a set $\{z_j\}_{j=1}^{\infty}$ can be the zero set of an $f \in H^p$ if and only if $\Sigma(1 - |z_j|) < \infty$. Notice that the condition does *not* depend on p. In striking contrast we have

THEOREM 7 (RUDIN). *Let $0 < p_0 < \infty$, $n > 1$. There is a set $Z \subseteq B$ such that Z is the zero set of an $f \in H^{p_0}(B)$, but is not the zero set of any $f \in H^p$, $p > p_0$.*

The proof, as is the case with many of those in the book, consists of ingenious use of purely classical techniques (Hardy and Littlewood knew the necessary tools). While the ball is very special, it cannot be denied that Theorem 7 provides important information to those who want to explore more general domains.

To my mind, Rudin has been the principal figure among those who have applied one-variable techniques to discover new phenomena in $\mathbf{C}^n$. This book is in many ways a catalogue of his results and those of his collaborators and followers. That is both an advantage and a disadvantage: on the one hand the book collects a lot of material which is scattered throughout the literature and provides an extensive and thoroughly researched bibliography; on the other hand it has the disjointed features of a catalogue, does not develop to any particular climax, does not have any unifying theme (except for the ball itself). In my view the book would be inappropriate for a first course in several complex variables. However it would be a fine choice for a second course, for a portion of a complex analysis or functional analysis course, or for a "topics" seminar. It will also be a useful reference, or a nice book to dip into from time to time.

Since Rudin dreams up such excellent exercises, it is too bad that he did not do so in this book. Several complex variables is a subject in which the student has much trouble getting started; Rudin's book will help alleviate the problem, but exercises would help a great deal. I also wish that some of the theorems were not formulated in such generality. For instance, Rudin's formulation of Theorem 7 is much more general than my own and for that reason loses a lot of its impact.

The book's positive features clearly majorize the few negative ones: it is lucid and accessible, it is written by the right person at the right time, and it will acquaint a lot of new (and heretofore apprehensive) people with a rapidly growing subject. It will probably contribute to another nine-fold increase in §32 of the Math. Reviews by the year 1999.

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Descriptive set theory, by Yiannis N. Moschovakis, Studies in Logic and the Foundations of Mathematics, vol. 100, North-Holland Publishing Company, Amsterdam, 1980, xii + 637 pp., $73.25.

THE BRAVE NEW WORLD OF DETERMINANCY

The birth of descriptive set theory was marked by one of those curious events that occasionally act as a catalyst for an important discovery. An error found by a twenty year old student in a major work by a famous mathematician started a chain of theorems leading to the development of a new mathematical discipline.

For the background on the beginnings of the theory of analytic and projective sets let us go back to the early years of this century, to France, where Messrs. Baire, Borel and Lebesgue were laying foundations of modern function theory and integration [**5**, **2**, **16**, **6**]. A real-valued function of several real variables is a *Baire function* if it belongs to the smallest class of functions which contains all continuous functions and is closed under the taking of pointwise limits. A set in real n-space is a *Borel set* if it belongs to the smallest class of sets which contains all open sets and is closed under the taking of countable unions and intersections. Baire functions form a hierarchy, indexed by countable ordinal numbers: functions of Baire class 0 are the continuous functions, functions of Baire class 1 are the limits of sequences of continuous functions, and in general, functions of Baire class ξ are the limits of sequences of functions belonging to Baire classes smaller than ξ. Borel sets are similarly arranged in a hierarchy: sets of Σ-class 1 ($\mathbf{\Sigma}_1^0$) and Π class 1 ($\mathbf{\Pi}_1^0$) are, respectively, the open sets and the closed sets, and for each countable ordinal ξ, the class $\mathbf{\Sigma}_\xi^0$ (the class $\mathbf{\Pi}_\xi^0$) consists of all countable unions (countable intersections) of sets belonging to classes smaller than ξ. There is an intimate relationship between the hierarchies of Baire functions and of Borel sets; this relationship was extensively studied by Lebesgue in [**17**]. For instance, a