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A first course in rational continuum mechanics, vol. I, by C. Truesdell, Pure and Applied Mathematics, Academic Press, New York, San Francisco, London, 1977, xxiii + 280 pp., $23.00.

The cornerstones of modern research into problems of nonlinear continuum mechanics can be said to have been laid in the late forties and early to mid-fifties by R. S. Rivlin and W. Noll. In the period following the end of the Second World War, Rivlin exploited the fact that internal constraints present in a nonlinearly elastic body, such as incompressibility, reduce the class of possible motions of such a body but greatly expand the set of stress fields which are compatible with such motions as may occur. Noting that the pressure field in an incompressible elastic body is not determined by the deformation of that body, and thus may be adjusted so as not only to help satisfy the boundary conditions, but also the equations of motion, Rivlin was able, in a rather short period of time, to produce exact solutions for a varied class of boundary value problems of nonlinear elasticity, i.e., torsion of an elastic circular cylinder, eversion of an incompressible elastic circular cylinder, etc. Reprints of several of Rivlin's pioneering papers may be found in the series of volumes **[1]** edited by Professor Truesdell; also included in this series are reprints of several of the early foundation papers of Walter Noll which appeared during the period 1955–58. In these latter papers Noll approaches the problem of formulating constitutive theories for elastic solids, fluids, and (the more general) simple materials with fading memory so as to be consistent with the principle of material frame indifference; essentially his arguments, which are now an integral part of modern continuum mechanics, serve to delimit those functions which may enter a hypothetical class of constitutive equations, and hence lead to reduction arguments that produce certain specific normalized forms for elastic and fluid materials as well as more general bodies which exhibit a precise sense of fading memory. Further reduction arguments based on the nature of the symmetry (or peer) group of a particular material, as well as the corresponding precise definitions of such commonly used (and abused) terms as solid, fluid, isotropic body, etc., also make their appearance for the first time, in a completely rigorous manner, in Noll's early work during the mid fifties.

Since the pioneering work of Rivlin and Noll there has been, in the last two and one-half decades, an explosion of work in the general area of modern continuum mechanics. Aside from the studies of new models of continua, much recent work has been directed into two major areas. First of all, considerable effort has been expended in trying to impart a concrete basis to the science of thermodynamics (at least in as much as to try to put it on an equal footing with classical mechanics and modern continuum mechanical theories of solids and fluids); a lucid account of the current state of affairs in this area may be found in the article by Serrin **[2]** and the recent texts by Truesdell **[3, 4]**; one central notion in almost all the current researches into the foundations of continuum thermodynamics is the interplay between formulations of the second law of thermodynamics and various concepts of stability; the article of Gurtin **[5]** is particularly illuminating with regard to work in this area. A second major area of work within the realm of modern continuum mechanics, and one which should have a broader appeal for workers in analysis (i.e., the theory of partial differential equations) concerns the ongoing effort to prove reasonable existence theorems for boundary and initial-boundary value problems associated with those equilibrium and evolution equations which arise in nonlinear elasticity. We will make a few general comments here about the nature of the problems involved.

Let χ_κ be a motion of an elastic body $\mathfrak{B}$ referred to the reference configuration κ; thus $\kappa(\mathfrak{B})$ is a bounded open connected set in R^3, $\chi_\kappa \in C^2(\kappa(\mathfrak{B}) \times R)$, and the present position $\mathbf{x}$ at time t of a particle X which occupies the place $\mathbf{X}$ in $\kappa(\mathfrak{B})$ is given by

$$\mathbf{x} = \chi_\kappa(\mathbf{X}, t), \qquad \mathbf{X} \in \kappa(B), \qquad t \in R.$$

Saying that the material body is elastic and homogeneous means that the gradient $\nabla\chi_\kappa$ is sufficient to determine the stress at each particle X and that the Piola-Kirchhoff stress tensor is given by

$$\mathbf{T}_\kappa = \mathfrak{F}_\kappa(\nabla\chi_\kappa)$$

where the response function $\mathfrak{F}_\kappa$ relative to κ is independent of $\mathbf{X}$; the particular type of elastic material being modelled depends on the choice of $\mathfrak{F}_\kappa$. (As a consequence of the principle of material frame-indifference it follows that if $\nabla\chi_\kappa = \mathbf{RU}$, $\mathbf{R}$ orthogonal and $\mathbf{U}$ positive definite then in fact $\mathfrak{F}_\kappa(\nabla\chi_\kappa) = \mathbf{R}\mathfrak{F}_\kappa(\mathbf{U})$.) From the expression for the stress tensor and Cauchy's First Law of Motion (balance of momentum) one then obtains the differential equation

$$\partial_t^2\chi_\kappa = \frac{1}{\rho_\kappa}\operatorname{Div}\mathfrak{F}_\kappa(\nabla\chi_\kappa), \qquad \mathbf{X} \in \kappa(B),$$

where ρ_κ is the mass density of $\mathfrak{B}$ in the reference configuration κ (reference "placement" in the newer terminology) and where zero external body force has been assumed. These equations may be written in (Cartesian) component form as

$$\partial_t^2\chi_k = \frac{1}{\rho_\kappa}A_{km}^{\alpha\beta}(\nabla\chi_\kappa)\chi^m_{,\alpha\beta}, \qquad k = 1, 2, 3,$$

where the

$$A_{km}^{\alpha\beta}(\mathbf{F}) = \frac{\partial\mathfrak{F}_k^\alpha(\mathbf{F})}{\partial F_\beta^m}, \qquad k, \alpha, m, \beta = 1, 2, 3,$$

for any invertible tensor argument $\mathbf{F}$, are the elasticities. In static (or equilibrium) problems such as the eversion problem for an elastic cylindrical tube which is discussed in **[6]** the differential system above reduces to

$$A_{km}^{\alpha\beta}(\nabla\chi_\kappa)\xi^m_{,\alpha\beta} = 0, \qquad k = 1, 2, 3,$$

and the typical zero traction boundary condition reads

$$\mathfrak{T}_k^\alpha(\nabla\chi_\kappa(\mathbf{X}))n_\alpha = 0, \qquad \mathbf{X} \in \partial\kappa(B)$$

where the n_α are the components of the unit exterior normal to $\partial\kappa(B)$ at $\mathbf{X}$. One of the unpleasant facts of life concerning the solution of boundary-value problems for the equilibrium equations associated with nonlinear elastic response is the necessity of allowing for physically, and accounting for mathematically, the existence of arbitrarily many solutions or equilibrium states. (See the illuminating discussion and photographs in [**6**, pp. 511–521].) To this end many workers in finite elasticity have searched for the right type of inequalities or restrictions to impose on the relevant constitutive functions, restrictions under which one could prove an existence theorem which allows for the possibility of bifurcation and the existence of multiple equilibrium states, i.e., one which does not at the same time yield unqualified uniqueness. Thus restrictions such as those resulting from assuming that $\mathfrak{T}_\kappa$ is a monotone mapping, i.e.,

$$(\mathbf{F} - \bar{\mathbf{F}}) \cdot \left[\mathfrak{T}_\kappa(\mathbf{F}) - \mathfrak{T}_\kappa(\bar{\mathbf{F}})\right] > 0, \qquad \mathbf{F} \neq \bar{\mathbf{F}},$$

or that $\mathfrak{T}_\kappa$ is such as to make every placement of the body infinitesimally stable, are of no value to the analyst here. On the other hand, the condition of strong ellipticity of the equations, namely,

$$A_{km}^{\alpha\beta}\lambda^k\lambda^m\mu_\alpha\mu_\beta > 0, \qquad \boldsymbol{\lambda} \neq \mathbf{0}, \quad \boldsymbol{\mu} \neq \mathbf{0},$$

has been shown to suffice for unqualified uniqueness of smooth solutions to the initial-boundary value problem of place, where χ_κ is prescribed on $\partial\kappa(B)$, while uniform strong ellipticity has been proven sufficient for the initial-value problem (for $\kappa(B) = R^3$) to be well posed in a sufficiently small time-interval. Thus under the assumption of strong ellipticity it is possible that there exist (other) nonsmooth solutions or smooth solutions which break down after a finite time. Indeed, for a large class of one-dimensional problems of nonlinear elasticity (elastic rods and shells), S. Antman [**1**] has used the strong ellipticity condition to prove the existence of classical solutions; his results allow for the commonly observed phenomena of buckling, necking, and shear instability that would be eliminated by unqualified uniqueness. No existence theorem of the desired type, however, has ever been proved for three-dimensional problems which is based on either the condition of strong ellipticity or the much discussed Coleman-Noll inequality (which implies a restricted type of uniqueness and allows for various bifurcation phenomena.) Quite recently, however, J. Ball [**8**, **9**] has imposed conditions on the response function $\mathfrak{T}_\kappa$, conditions such as "polyconvexity" (a special manifestation of the concept of quasiconvexity introduced by Morrey [**10**]), has related these conditions to those of ellipticity and strong ellipticity, and has obtained an existence theorem which does not imply unqualified uniqueness of solutions; the solutions whose existence is proven in Ball's work are generalized solutions of the equilibrium equations, and thus the regularity theory still must be developed. Indeed the rigorous study of existence, uniqueness, stability, and bifurcation for three-dimensional problems in nonlinear theories of elastic and solid phenomena is an area in which many talented analysts are working today, and the above discussion has touched upon the work that is ongoing in only one very small corner of a vast arena.

The book currently under review does not, of course, cover either of the two major areas of current research referred to above. In a second volume on elasticity and fluid dynamics, a detailed table of contents for which is included in the present volume, the author indicates his clear intention to treat some of the mathematical problems of nonlinear elasticity we have discussed while a third volume (listed as being tentative) will no doubt discuss some of the problems connected with formulating a rigorous theory of continuum thermodynamics (as well as exploring some of the relations between thermodynamics and dynamic stability theories). The present volume, the first in a proposed set of three, is intended as a textbook and deals, in successive chapters, with

I. Bodies, forces, motions, and energies,

II. Kinematics,

III. The stress tensor,

IV. Constitutive relations,

An appendix sketches certain basic results and theorems of algebra, geometry, and differential calculus; another appendix contains solutions to many of the exercises that appear in the body of the text itself. Thus this first volume discusses concepts which pertain to all theories of continuous media regardless of whether they be solid or fluid, elastic or something more general then elastic, although with a view in mind to volume II, most of the examples and illustrations concern either elastic or fluid bodies. As for the text itself, the first chapter presents a rigorous and extremely detailed axiomatic foundation for the concepts of body, force, motion, and energy as they relate to modern continuum theories; the axiomatic formulation is based upon quite recent work of Noll, Gurtin, et. al., and will, in this reviewer's opinion, provide the reader with a forest of ideas and abstract concepts from which he may find it difficult to escape, especially if he is new to the subject as a whole. Indeed, the author himself seems to sense this when he states (p. 5, §1) that "the presentation...is abstract. The reader who is content to take bodies, the event world, frames of reference, motions, forces, and energies for granted may skip this chapter and pass to the next one, which begins the formal treatement of continuum mechanics along traditional lines". While I do not feel it advisable to "skip this chapter" I think that a light reading, hitting only the high points, is to be advised for the novice, regardless of his mathematical training; indeed, someone using this book as a text for a class being initiated into the concepts of modern continuum mechanics might prefer to do just that and, perhaps, would have preferred to see a much shorter first chapter, one which introduced only the basic ideas and definitions (the minimum necessary to understand Chapters II–IV) with the remainder of the material relegated to an appendix. The remaining three chapters are standard and follow the traditional lines laid down in earlier works by Professor Truesdell (this reviewer is particularly partial to the nice, compact treatment in **[11]**). Here, fortunately, many of the details which could be passed over lightly in a first reading have been relegated to the many exercises but not always, e.g., the discussion of mean values and lower bounds for stress fields in §7 of Chapter III, the discussion of the motion of a free body in §9 of the same chapter, etc., are among topics that one might want to sacrifice on a first go through if, indeed, the book is to be used for a text and one is constrained, as need be in a formal course, by limitations of time (usually two semesters to cover the contents of volumes I–III of the proposed set). Perhaps, the whole volume could have been well served by having had a guide or suggested outline of sections to be covered (in a first reading or course) included. The discussion of Kinematics in Chapter II closely follows the pattern of the relevant parts of the encyclopedic treatise by the author and R. A. Toupin [**12**], while the treatment of the stress tensor in Chapter III and the treatment of the reduction and classification of constitutive theories in Chapter IV, employing material frame indifference and the peer (material symmetry) group, closely follows the related treatment in the other encyclopedic treatise by Truesdell and Noll [**13**]. The material in Chapters II–IV has, of course, been considerably updated in terms of notation, terminology, and results as compared with the corresponding material in [**12, 13**]. This reviewer laments the lack of a comprehensive bibliography for the text, one which would contain references to the relevant works of the many authors whose names are attached to the theorems throughout the volume (especially since many of these theorems are of a recent vintage and concern ongoing work on problems which are still of current research interest). An enormous plus to the text is the large number of solved problems at the end of the book. In summarizing we may conclude that the volume is masterfully written and remarkably devoid of misprints and will with its companion volumes II and III be a welcome addition to the growing literature on continuum mechanics; because of the scope of the work (the three volumes taken together may be almost as encyclopedic in nature as either of the works [**12, 13**]) and the lack of a detailed guide for the prospective student, one would have to hope that within the context of a formal course, it will be used by an instructor who has some distinct familiarity with the subject matter itself as it has developed over the last two to three decades.

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CORRIGENDUM

The theory of recursive functions, approaching its centennial, by Stephen C. Kleene, this Bulletin, vol. 5, 1981, p. 43

FOOTNOTE NUMBER 1. In the second line the name of Robert I. Soare should be substituted for that of Richard A. Shore as organizer of four special sessions.