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Basic theory of algebraic groups and Lie algebras, by Gerhard P. Hochschild, Graduate Texts in Mathematics, vol. 75, Springer-Verlag, New York, Heidelberg, Berlin, 1981, viii + 267 pp., $32.00. ISBN 0-3879-0541-3

Linear algebraic groups, by T. A. Springer, Progress in Mathematics, vol. 9, Birkhauser, Boston; Basel, Stuttgart, 1981, x + 304 pp., $20.00. ISBN 3-7643-2039-5

A unified theory of linear algebraic groups emerged only in the 1940s. Before that time, special classes of algebraic groups such as the orthogonal groups and the general linear groups had been carefully studied, but these were often viewed separately and independently rather than as parts of some greater whole. In his *Théorie des groupes de Lie* [7], Chevalley laid the foundations for this more general theory. He developed the subject in the spirit of classical Lie theory by associating to each group its Lie algebra and by utilizing a formal exponential mapping from the Lie algebra to the group. Unfortunately, this process of linearization only worked well when the base field was of characteristic zero. At least one important result, the Lie-Kolchin theorem, did hold for groups in arbitrary characteristic. It stated that a connected solvable linear algebraic group defined over an algebraically closed field was conjugate to a group of upper triangular matrices. (See Kolchin [**13**].)

By the 1950s, however, the systematic use of developments in algebraic geometry began to supplant the traditional Lie-theoretic approach to algebraic groups. This infusion of new techniques made possible not only the construction of a theory which was valid in arbitrary characteristic, but also the discovery of new results which had previously been unknown even in characteristic zero. The celebrated Borel fixed-point theorem provided a prime example of this phenomenon. Borel showed [**1**] that a connected solvable linear algebraic group (over an algebraically closed field) which acted on a complete variety (in the appropriate sense) had a fixed-point. When applied to the flag variety of a vector space on which the group acted linearly, this result yielded a new algebraic geometric proof of the above-mentioned Lie-Kolchin theorem.

Making use of this new approach, Chevalley succeeded in completely classifying the simple algebraic groups over an algebraically closed field k. His attack was two-pronged. He showed [**5**] that two simple groups over k with isomorphic root systems and weight lattices were isomorphic. More generally, he determined the isogenies between two simple (or semisimple) groups. In his great "Tôhoku" paper [**6**], however, he actually carried out a construction of the simple (adjoint) groups. This process, which began with the corresponding complex simple Lie algebras, also gave rise to the finite Chevalley groups and, a little later, to the important finite groups of Lie type. (See Carter [**4**] for a treatment of this aspect.) Amazingly enough, in spite of the substantial recasting and generalization, the new classification perfectly mirrored its classical characteristic zero analogue.

After Chevalley's discovery of the classification theorems, algebraic group theory grew and prospered both as an area of mathematics in its own right and as a useful tool in such fields as finite groups, algebraic geometry, and automorphic forms. (See Borel's article [**3**] for a discussion of these matters.) As a testimony to the subject's coming of age, introductory accounts began to appear in the late 1960s and 1970s, most notably Borel's *Linear algebraic groups* [**2**] (1969) and Humphreys' book [**12**] (1975) of the same title. Most recently, G. P. Hochschild and T. A. Springer have each made important contributions to this growing stock of introductory literature.

Hochschild sets the tone of his book in the preface when he states that the "...emphasis is on developing the major general mathematical tools used for gaining control over algebraic groups, rather than on securing final definitive results..." (p.v.). His two most powerful tools, algebraic geometry and the theory of Lie algebras, demand quite a bit of attention. In fact, Hochschild develops these topics to the extent that his treatments could serve as introductions for the uninitiated novice.

The discussion of algebraic geometry he gives here closely follows that found in Humphreys' [**12**], but Hochschild departs from Humphreys' sketch by meticulously including every proof. Primarily, he spares no effort in dealing with the pertinent areas of commutative algebra. For example, he thoroughly develops both the dimension theory of local rings and the structure theory of regular local rings. In discussing Lie algebras, the author reproduces the polished development he gave in **[11]** to obtain such classical results as Cartan's solvability criterion and Weyl's complete reducibility theorem. He finishes up with the canonical presentation of the basic structure theory as well as the representation theory of semisimple Lie algebras.

With these results and techniques at his disposal, Hochschild describes the two main slants on the theory of linear algebraic groups. He gives expositions both of the general structure theory over arbitrary algebraically closed fields and of the more classical characteristic zero theory. His handling of the arbitrary field case proceeds along traditional lines and concludes with the theory of Borel subgroups. He makes full use of algebraic geometry in obtaining such results as the existence of quotient varieties and the Borel fixed-point theorem. As for the Lie algebra theory, he calls this into service to prove more incisive results about the groups in characteristic zero. Some of this comes from Chevalley **[7]**, of course, but Hochschild includes much more. We find, for example, a proof of Mostow's semidirect product theorem **[14]** not to mention the masterful closing chapter where Hochschild presents an elegant construction (based on **[10]**) of the simply connected groups having Lie algebras with nilpotent radicals. This is definitely the high point of the book. The author's treatment of the characteristic zero theory is a tour de force.

In spite of these strengths, this book will not appeal to everyone. While the author succeeds in achieving his announced purpose of developing the fundamental tools, the lack of direction and the rarity of clarifying examples represent potential stumbling blocks for the beginner. A glance at the table of contents reveals a medley of topics from "Representative Functions and Hopf Algebras" to "Derivations and Lie Algebras," then from "Algebraic Varieties" to "Semisimple Lie Algebras". In this last chapter there is neither an example of a semisimple Lie algebra nor one of a specific irreducible representation, although these are the objects under investigation.

The more experienced reader, however, will appreciate Hochschild's book for its concise and authoritative style as well as for its treatment of various topics not usually found in introductory texts. We point out, in addition to the final chapter mentioned above, his discussion of automorphism groups and observable subgroups. In summary, this book is a rich and potentially useful treatment of algebraic groups. The same may also be said of T. A. Springer's new text on the same subject.

Based on lectures given at the University of Notre Dame in 1968, Springer's book provides a complete and essentially self-contained account of the classification of reductive (and hence simple) algebraic groups over a general algebraically closed field k. In contrast to the traditional approach, Springer presents a new construction of these reductive groups which makes no use of the classical Lie algebra theory. Thus, he limits his preliminary material to basic algebraic geometry. In about fifty pages and with a minimal amount of fuss, the reader acquires all of the tools needed for the task ahead.

First, Springer tackles the general structure theory. Here he follows the usual outline, but his proofs are often new and improved. The development culminates in Chapters 9 and 10 where he associates to each reductive group its "root datum" $(R, X, R^\vee, X^\vee)$ in the sense of [**8**] and then proves such fundamental results as the Bruhat decomposition lemma.

Next, in Chapter 11, he carries out half of Chevalley's classification program by showing that two reductive groups over k with isomorphic root data are isomorphic. The novel argument comes in several parts. Given a reductive group G with root system R, Springer suitably normalizes the usual one-parameter unipotent subgroups x_α $(\alpha \in R)$ of G. This gives rise to "structure constants" for G via Chevalley's commutator formula. By giving a "generators and relations" presentation of G, he shows that these constants are determined (up to a natural equivalence) by the root datum of G. The desired result then follows straightaway with the existence of graph automorphisms as an easy corollary. Springer concludes this train of thought with a brief sketch of Chevalley's isogeny theorem.

The coup de grâce comes in Chapter 12 when the author proves the existence of a reductive group having a prescribed root datum. Reducing to the special case of a semisimple group G of adjoint type, he presents G as an automorphism group of a Lie algebra, provided R is simply laced. His slick argument in constructing the Lie algebra hinges on a device first used by Frenkel and Kac [**9**]. Finally, in case R is not simply laced, he obtains G from the fixed-points of a suitable graph automorphism.

This sharp treatment of algebraic groups should become the standard reference for classification theory. Informal and engaging in style, the work makes for pleasant reading. It is further enhanced by the presence of numerous examples and timely exercises on such topics as Schubert varieties and Bruhat orderings in Weyl groups. The inordinate number of typographical errors, however, definitely detracts from its overall appeal, although it presents no real problem.

Despite their common ground, these two books are remarkably different. For those interested primarily in the characteristic zero theory, Hochschild's work should prove invaluable, whereas both books should benefit the enthusiast of the general theory. Each in his own way, Springer and Hochschild have filled gaps in the literature.

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