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Combinatorial enumeration, by Ian P. Goulden and David M. Jackson, John Wiley & Sons, Inc., Somerset, New Jersey, 1983, xxiv + 569 pp., $47.50. ISBN 0-4718-6654-7

The most important idea in enumerative combinatorics is that of a generating function. According to the classical viewpoint, if the function $f(x)$ has a power series expansion $\sum_{n=0}^{\infty} a_n x^n$, then $f(x)$ is called the generating function for the sequence a_n. Sometimes the coefficients b_n, defined by

$$f(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!},$$

are more useful; here $f(x)$ is called the exponential (or factorial) generating function for the sequence b_n. Generating functions are often easier to work with than explicit formulas for their coefficients, and they are useful in deriving recurrences, congruences, and asymptotics.

Although generating functions have been used in enumeration since Euler, only in the past twenty years have theoretical explanations been developed for their use. Some of these, such as those of Foata and Schützenberger [**6, 7**] and Bender and Goldman [**2**] use decompositions of objects to explain generating function relations. Other approaches, such as those of Rota [**13**], Doubilet, Rota, and Stanley [**4**], and Stanley [**14**], use partially ordered sets. Goulden and Jackson's book is a comprehensive account of the decomposition-based approach to enumeration.

In the classical approach to generating functions, one has a set A of configurations (for example, finite sequences of 0's and 1's) satisfying certain conditions. Each configuration has a nonnegative integer "length". The problem is to find the number a_i of configurations of length i. One first finds a recurrence for the a_i by combinatorial reasoning; the recurrence then leads to an algebraic or differential equation for the generating function. The generating function is viewed as a formal device with no inherent meaning, and the choice of an ordinary or exponential generating function is pragmatic: one or the other will be easier to work with. (Sometimes an exponential generating function is explained as necessary for convergence.)

In the modern approach to generating functions, each element of the set A of configurations to be counted is assigned a "weight". The generating function for A is simply the sum of the weights of its elements. In most applications the weights are products of indeterminates, so if A is infinite, the sum is a formal power series. Convergence of this power series to an analytic function of one or several variables is irrelevant, and thus the term "generating function" is somewhat misleading (but seems unlikely to be replaced by the more accurate alternative term "counting series"). Properties of the generating function often follow directly from the structure of A.

Let us look at a very simple problem from the classical and modern points of view. Let a_n be the number of compositions of n with parts 1 and 2, that is, the number of ways of expressing n as an ordered sum of 1's and 2's. Thus $a_3 = 3$, since $3 = 1 + 2 = 2 + 1 = 1 + 1 + 1$. In the traditional approach we observe that a composition of $n > 0$ consists either of a composition of $n - 1$ followed by a 1 or a composition of $n - 2$ followed by a 2. Thus we have the recurrence $a_n = a_{n-1} + a_{n-2}$ for $n > 0$, with the initial conditions $a_{-1} = 0$, $a_0 = 1$. If we set $a(x) = \sum_{n=0}^{\infty} a_n x^n$, then multiplying the recurrence by x^n and summing on $n \geqslant 1$ yields $a(x) - 1 = a(x)(x + x^2)$, so

$$a(x) = 1/(1 - x - x^2).$$

In the modern approach we take A to be the set of compositions of nonnegative integers with parts 1 and 2. We define the weight of a composition of n to be x^n, where x is an indeterminate. If we define the weight of a 1 to be x and the weight of a 2 to be x^2, then the weight of a composition is the product of the weights of its parts. Thus by considering the last part of a nonempty composition we find a weight-preserving bijection $A - \{\varnothing\} \simeq A \times \{1, 2\}$. Taking the sum of the weights on both sides, we obtain $a(x) - 1 = a(x) \cdot (x + x^2)$, as before. This is an example of what Goulden and Jackson call a *recursive decomposition*.

There is a slightly different way of deriving the generating function, which is important. Let A_k be the set of compositions with k parts. Then there is a weight-preserving bijection from A_k to $\{1, 2\}^k$, so the generating function for A_k is $(x + x^2)^k$. Thus, summing on k, we find the generating function for A to be

$$\sum_{k=0}^{\infty} (x + x^2)^k = 1/(1 - x - x^2).$$

This is a *direct decomposition*.

By exactly the same reasoning one can find a generating function for compositions with any specified set of parts weighted arbitrarily.

Similar, but more complicated, decompositions can be used to count certain types of trees. A rooted binary plane tree consists either of a single vertex (the root) or of a root together with an ordered pair of trees (its left and right subtrees). Thus if T is the set of rooted binary plane trees and R is the set containing only a single tree with one vertex, we have a bijection $T - R \simeq R \times T \times T$. If we assign a tree with n nodes the weight x^n, then the generating function $t(x)$ for trees satisfies $t(x) - x = xt^2(x)$, which can be solved to give

$$t(x) = \frac{1 - \sqrt{1 - 4x^2}}{2x} = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^{2n+1}.$$

More generally, the generating function for rooted k-ary plane trees satisfies $t(x) - x = xt^k(x)$. These equations can be solved by the Lagrange inversion formula.

Exponential generating functions, which are of the form $\sum_{n=0}^{\infty} a_n x^n/n!$, arise in counting "tagged" (or "labeled") configurations: a_n is the number of objects of some type which can be made from the set of tags $\{1, 2, \ldots, n\} = N_n$. Examples are permutations of N_n and labeled graphs on N_n. (For a more precise category-theoretic approach to "tagged configurations" see Joyal **[10]**.) The usefulness of exponential generating functions arises from the fact that

$$\frac{x^m}{m!} \cdot \frac{x^n}{n!} = \binom{m+n}{m} \frac{x^{m+n}}{(m+n)!},$$

where the binomial coefficient $\binom{m+n}{m}$ is the number of m-element subsets of an $(m+n)$-element set. If A is a configuration with tag-set N_m and B is a configuration with tag-set N_n, we can combine them in $\binom{m+n}{m}$ ways to get a configuration $(\bar{A}, \bar{B})$ with tag-set N_{m+n}: We first choose an m-element subset S of N_{m+n} and replace the tags of A with the elements of S (preserving their order) to get $\bar{A}$, and in the same way we get $\bar{B}$ from B and $N_{m+n} - S$.

Thus if $f(x)$ and $g(x)$ are exponential generating functions for classes of tagged configurations, their product $f(x)g(x)$ will be the exponential generating function for ordered pairs of these configurations. For example,

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

is the exponential generating function for nonempty sets. Thus

$$(e^x - 1)^2 = \sum_{n=2}^{\infty} (2^n - 2) \frac{x^n}{n!}$$

is the exponential generating function for ordered partitions of a set into two nonempty blocks.

Not only can we add and multiply exponential generating functions, we can also compose them. If $f(x)$ is the exponential generating function for certain tagged configurations, then $f(x)^k$ counts k-tuples of these configurations. Now if $f(0) = 0$, so that every configuration has at least one tag, then $f(x)^k/k!$ counts sets of k of these configurations. Thus, for example, $(e^x - 1)^k/k!$ counts partitions of a set into k (nonempty) blocks and $\exp(e^x - 1)$ counts all partitions of a set. In general, $e^{f(x)}$ will count sets of configurations, each counted by $f(x)$.

One of the first applications of exponentiation of exponential generating functions was to counting connected labeled graphs (Riddell and Uhlenbeck [**12**]). A graph on n vertices is a subset of the set of $\binom{n}{2}$ pairs of vertices, so there are $2^{\binom{n}{2}}$ graphs on n vertices. A graph may also be identified with the set of its connected components, so if $c(x)$ is the exponential generating function for connected graphs,

$$\sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!} = e^{c(x)}.$$

This is an example of an *indirect decomposition*. "Known" configurations are decomposed into the configurations to be counted, yielding a generating function relationship that can be solved for the unknown generating function.

Goulden and Jackson's book uses decompositions of the kind described above to derive most of the classical results of enumerative combinatorics. In addition to the standard topics, their book includes much that is not well known. Most of the material has not appeared in books before, and some of it is completely new. We now discuss the contents chapter by chapter.

Chapter 1 covers properties of formal power series in several variables that are useful in enumeration, including the multivariable Lagrange inversion formula. Chapters 2 and 3 are devoted to ordinary and exponential generating functions and cover most of the basic material of enumeration, together with some little-known applications. Chapter 2 begins with the elementary counting lemmas relating combinatorial decompositions to generating functions, and applies them to problems involving sequences, partitions, plane trees, and planar maps. Chapter 3 covers familiar topics, such as labeled trees and cycle structures of permutations, and some unusual ones, such as bicoverings of a set.

The last two chapters are devoted to more advanced topics. Chapter 4 is concerned with "sequence enumeration" problems. Many of these count permutations with a given pattern of rises and falls. The oldest result of this type is D. André's [**1**], which states that the number of permutations $a_1 a_2 \cdots a_n$ of N_n satisfying $a_1 < a_2 > a_3 < a_4 \cdots$ is the coefficient of $x^n/n!$ in $\sec x + \tan x$. Closely related is Simon Newcomb's problem, first solved by MacMahon [**11**], which asks for the number of arrangements $a_1 a_2 \cdots a_m$ of k_1 1's, k_2 2's,..., such that $a_i > a_{i+1}$ for a given number of i's. A different type of permutation problem is the ménage problem: How many ways can n couples sit around a circular table with men and women alternating, but no husband and wife adjacent?

Some permutation problems (including Simon Newcomb's problem) were treated in Chapter 2 by fairly elementary methods. In Chapter 4 powerful techniques, due to the authors, are developed for solving these problems with matrices.

Chapter 5 discusses four recent topics involving paths: Flajolet's combinatorial theory of continued fractions [**5**], a factorization method for counting paths restricted to a half-plane [**8**], the enumeration of nonintersecting paths, and a q-analogue of the Lagrange inversion formula which is proved via paths [**9**].