

SINGULAR LOCI OF SCHUBERT VARIETIES FOR CLASSICAL GROUPS

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In this note, we give an explicit description of the singular locus of a Schubert variety in the flag variety G/B , where G is a classical group, and B a Borel subgroup of G .

Let G be a classical group, and T a maximal torus in G . Let W be the Weyl group, and R the system of roots, of G relative to T . Let B be a Borel subgroup of G , where $B \supset T$. Let S (resp. R^+) be the set of simple (resp. positive) roots of R relative to B . For $\alpha \in R$, let s_α be the reflection with respect to α , and X_α the element in the Chevalley basis for the Lie algebra of G , associated to α . For $w \in W$, let $e(w)$ denote the point in G/B corresponding to w . The Schubert variety $X(w)$, where $w \in W$, is by definition the Zariski closure of $B e(w)$ in G/B . ($X(w)$ is understood to be endowed with the canonical reduced structure.) Let \succeq denote the Bruhat order in W . It is well known that for $w_1, w_2 \in W$,

$$w_1 \succeq w_2 \quad \text{if and only if} \quad X(w_1) \supseteq X(w_2).$$

(For generalities on algebraic groups, one may refer to [1].)

The results on the singular locus of a Schubert variety are obtained as consequences of "standard monomial theory" as developed in *Geometry of G/P* . I-V (cf. [11, 7, 4, 5, 8]). One of the consequences of standard monomial theory is the First Basis Theorem (cf. [5, 8, 6]) which gives a \mathbf{Z} basis

$$\{P(\lambda, \mu), (\lambda, \mu) \text{ an admissible pair}\}$$

for $H^0(G_{\mathbf{Z}}/P_{\mathbf{Z}}, L_{\mathbf{Z}})$, where $P_{\mathbf{Z}}$ is a maximal parabolic subgroup scheme of $G_{\mathbf{Z}}$ and $L_{\mathbf{Z}}$ is the ample generator of $\text{Pic}(G_{\mathbf{Z}}/P_{\mathbf{Z}})$. For any field k , let us denote the canonical image of $P(\lambda, \mu)$ in $H^0(G_{\mathbf{Z}} \otimes k/P_{\mathbf{Z}} \otimes k, L_{\mathbf{Z}} \otimes k)$ by $p(\lambda, \mu)$. In [9], it is shown that over any field k , for $w, \tau \in W$, with $w \succeq \tau$, the Zariski tangent space $T(w, \tau)$, to $X(w)$ at $e(\tau)$ is spanned by

$$\left\{ X_{-\beta}, \beta \in \tau(R^+) \mid \begin{array}{l} \text{for all } (\lambda, \mu) \text{ such that } X_{-\beta} p(\lambda, \mu) = c p(\tau, \tau), c \in k^*, \\ p(\lambda, \mu)|_{X(w)} \neq 0 \end{array} \right\}.$$

Denoting by $\{Q(\lambda, \mu)\}$ the basis for the \mathbf{Z} -dual of $H^0(G_{\mathbf{Z}}/P_{\mathbf{Z}}, L_{\mathbf{Z}})$, dual to the basis $\{P(\lambda, \mu)\}$, it can be seen easily that $X_{-\beta} p(\lambda, \mu) = c p(\tau, \tau)$, $c \in k^*$, if and only if $X_{-\beta} Q(\tau, \tau)$, when written as a \mathbf{Z} -linear combination of the elements

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$Q(\theta, \delta)$, involves $Q(\lambda, \mu)$ with a coefficient that is nonzero in k . From this we obtain that $T(w, \tau)$ is spanned by

$$\left\{ X_{-\beta}, \beta \in \tau(R^+) \left| \begin{array}{l} \text{for all } (\lambda, \mu) \text{ such that } Q(\lambda, \mu) \text{ occurs in } X_{-\beta}Q(\tau, \tau) \\ \text{with a coefficient that is nonzero in } k, w \succeq \lambda \end{array} \right. \right\}.$$

In [3], we have given an explicit description of $Q(\lambda, \mu)$ for the case of a classical group. Using this description, we express $X_{-\beta}Q(\tau, \tau)$ as a linear combination with integer coefficients of the $Q(\theta, \delta)$'s. This enables us to obtain an explicit description of the singular locus of $X(w)$.

Let G be classical of rank n . Let $S = \{\alpha_1, \dots, \alpha_n\}$, the order being as in [2]. Further, we follow the notation in [2] to denote the elements of R . For $1 \leq d \leq n$, we fix the following:

$$\begin{aligned} P_d &= \left\{ \begin{array}{l} \text{the maximal parabolic subgroup of } G \\ \text{obtained by "omitting the simple root } \alpha_d", \end{array} \right. \\ W_{P_d} &= \text{Weyl group of } P_d, \\ W^{P_d} &= \text{the set of "minimal representatives" of } W/W_{P_d}. \end{aligned}$$

Recall (cf. [2, 4]) that

$$W^{P_d} = \{w \in W \mid l(ws_{\alpha_i}) = l(w) + 1, 1 \leq i \leq n, i \neq d\}.$$

It is known (cf. [2]) that

$$(1) \quad W = W_{P_d} \cdot W^{P_d}.$$

For $w \in W$, let $w^{(d)}$ be the element in W^{P_d} corresponding to the coset wW_{P_d} . We have

$$(2) \quad w^{(d)} = wW_{P_d} \cap W^{P_d}.$$

Let

$$A = \{(a_1, \dots, a_d) \mid a_1 < a_2 < \dots < a_d, a_i \in \mathbf{Z}\}.$$

We have a natural partial order \geq in A , namely,

$$(3) \quad (a_1, \dots, a_d) \geq (b_1, \dots, b_d), \quad \text{if } a_i \geq b_i, 1 \leq i \leq d.$$

This partial order among d -tuples will be used in the sequel in describing the Bruhat order in W^{P_d} . Further, for any d -tuple (z_1, \dots, z_d) of integers, we let

$$(4) \quad (z_1, \dots, z_d) \uparrow = (z_{i_1}, z_{i_2}, \dots, z_{i_d})$$

where $j \rightarrow i_j$ is a permutation and $z_{i_j} \leq z_{i_{j+1}}$. Thus, $(z_1, \dots, z_d) \uparrow$ is the d -tuple whose entries are obtained by arranging the entries (z_1, \dots, z_d) in increasing order. We shall denote the elements of the symmetric group S_m , where $m \in \mathbf{N}$, in the following way. Let $\sigma \in S_m$ be such that

$$(5) \quad \sigma(i) = c_i, \quad 1 \leq i \leq m.$$

We shall denote σ by $(c_1 \dots c_m)$. Let k be the base field. For any positive integer m , let $\{e_1, \dots, e_m\}$ denote the standard basis of k^m .

I. The symplectic group $\mathrm{Sp}(2n)$. Let $E = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$, where

$$J = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}_{n \times n}.$$

Let $(,)$ be the skew symmetric bilinear form on k^{2n} , represented by E , with respect to $\{e_1, \dots, e_{2n}\}$. Let

$$(6) \quad G = \mathrm{Sp}(2n) = \{A \in \mathrm{SL}(2n) \mid {}^t A E A = E\}.$$

Let σ be the involution on $\mathrm{SL}(2n)$ defined by

$$(7) \quad \sigma(A) = E({}^t A)^{-1} E^{-1}, \quad A \in \mathrm{SL}(2n).$$

We see that

$$(8) \quad \mathrm{Sp}(2n) = \mathrm{SL}(2n)^\sigma.$$

In view of (8), we obtain an identification of W , the Weyl group of G , with a subgroup of S_{2n} (= the Weyl group of $\mathrm{SL}(2n)$), namely

$$(9) \quad W = \{(a_1 \cdots a_{2n}) \mid a_i = 2n + 1 - a_{2n+1-i}, 1 \leq i \leq 2n\}.$$

See [7] for details.

The above identification (cf. (9)) of W , and straightforward calculations using the definitions of [2] allow us to identify W^{P_d} as

$$(10) \quad W^{P_d} = \left\{ (a_1, \dots, a_d) \left| \begin{array}{l} (1) 1 \leq a_1 < a_2 < \cdots < a_d \leq 2n, \\ (2) \text{ for } 1 \leq i \leq 2n, \text{ if } i \in \{a_1, \dots, a_d\}, \\ \text{then } 2n + 1 - i \notin \{a_1, \dots, a_d\} \end{array} \right. \right\}.$$

For $w \in W$, say $w = (c_1 \cdots c_{2n})$, we see easily that

$$(11) \quad w^{(d)} = (c_1, \dots, c_d) \uparrow.$$

Under the above identification of W^{P_d} , we have (cf. [10]), given two elements $(a_1, \dots, a_d), (b_1, \dots, b_d)$ in W^{P_d} ,

$$(12) \quad (a_1, \dots, a_d) \succeq (b_1, \dots, b_d) \quad \text{if and only if} \quad (a_1, \dots, a_d) \geq (b_1, \dots, b_d).$$

Thus, the Bruhat order in W^{P_d} coincides with the natural order (cf. equation (3)) on d -tuples.

PROPOSITION C.1. *Let $G = \mathrm{Sp}(2n)$. For $1 \leq i \leq 2n$, let $i' = 2n + 1 - i$ and $|i| = \min\{i, i'\}$. Let $w, \tau \in W$, with $w \succeq \tau$. Let $\tau = (a_1 \cdots a_{2n})$. Then the tangent space $T(w, \tau)$ to $X(w)$ at $e(\tau)$ is spanned by the set of root vectors $\{X_{-\beta}, \beta \in N(w, \tau)\}$, where $N(w, \tau)$ is the subset of $\tau(R^+)$ consisting of roots β which satisfy criteria (a) and (b) below. Let $\beta = \tau(a)$, $\alpha \in R^+$. We follow the notation of [2] for elements of R^+ .*

(a) Let $\alpha = \varepsilon_j - \varepsilon_k$, $1 \leq j < k \leq n$ or $2\varepsilon_j$, $1 \leq j \leq n$. Then

$$w \succeq s_\beta \tau.$$

(b) Let $\alpha = \varepsilon_j + \varepsilon_k$, $1 \leq j < k \leq n$. Let s (resp. r) be the $\min\{|a_j|, |a_k|\}$ (resp. $\max\{|a_j|, |a_k|\}$). Then

$$w^{(j)} \succeq (a_1, \dots, a_{j-1}, a'_k) \uparrow$$

and

$$w^{(k)} \succeq (a_1, \dots, \hat{a}_j, \dots, a_{k-1}, r, s') \uparrow.$$

II. The special orthogonal group $\text{So}(2n+1)$. Let

$$E = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}_{2n+1 \times 2n+1},$$

and let $(\ , \)$ be the symmetric bilinear form on k^{2n+1} , represented by E , with respect to $\{e_1, \dots, e_{2n+1}\}$. Let

$$(13) \quad G = \text{So}(2n+1) = \{A \in \text{SL}(2n+1) \mid {}^t AEA = E\}.$$

Let σ be the involution on $\text{SL}(2n+1)$ defined by

$$(14) \quad \sigma(A) = E({}^t A)^{-1}E, \quad A \in \text{SL}(2n+1).$$

As in §I, we have

$$(15) \quad \text{So}(2n+1) = \text{SL}(2n+1)^\sigma.$$

In view of (15), we obtain identifications for the Weyl group W , and also for W^{Pa} similar to (9) and (10), namely

$$(16) \quad W = \{(a_1 \cdots a_{2n+1}) \in S_{2n+1} \mid a_i = 2n+2 - a_{2n+2-i}, 1 \leq i \leq 2n+1\}$$

and

$$(17) \quad W^{Pa} = \left\{ (a_1, \dots, a_d) \left| \begin{array}{l} (1) 1 \leq a_1 < a_2 < \cdots < a_d \leq 2n+1, \\ (2) a_i \neq n+1, 1 \leq i \leq d, \\ (3) \text{ For } 1 \leq i \leq 2n+1, \text{ if } i \in \{a_1, \dots, a_d\} \\ \text{ then } 2n+2-i \notin \{a_1, \dots, a_d\} \end{array} \right. \right\}.$$

For $w \in W$, say $w = (c_1 \cdots c_{2n+1})$, we have

$$(18) \quad w^{(d)} = (c_1, \dots, c_d) \uparrow.$$

As in §I, we have (cf. [10]) that the Bruhat order in W^{Pa} coincides with the natural order (cf. equation (3)) on d -tuples.

PROPOSITION B.1. (Assume $\text{char } k \neq 2$.) Let $G = \text{So}(2n+1)$. For $1 \leq i \leq 2n+1$, let $i' = 2n+2-i$ and $|i| = \min\{i, i'\}$. Let $w, \tau \in W$, with $w \succeq \tau$, and let $\tau = (a_1 \cdots a_{2n+1})$. Then the tangent space $T(w, \tau)$ to $X(w)$ at $e(\tau)$ is spanned by the set of root vectors $\{X_{-\beta}, \beta \in N(w, \tau)\}$, where $N(w, \tau)$ is the subset of $\tau(R^+)$ consisting of roots β which satisfy criteria (a), (b), and (c) below. Let $\beta = \tau(\alpha)$, $\alpha \in R^+$.

(a) Let $\alpha = \varepsilon_j - \varepsilon_k$, $1 \leq j < k \leq n$. Then

$$w \succeq s_\beta \tau.$$

(b) Let $\alpha = \varepsilon_j + \varepsilon_k$, $1 \leq j < k \leq n$. Let s (resp. r) be the minimum (resp. maximum) of $\{|a_j|, |a_k|\}$.

(i) Suppose precisely one of $\{a_j, a_k\}$ does not exceed n . Then

$$\begin{aligned} w^{(j)} &\succeq (a_1, \dots, a_{j-1}, a'_k) \uparrow, \\ w^{(k)} &\succeq (a_1, \dots, \hat{a}_j, \dots, a_{k-1}, r, s') \uparrow, \end{aligned}$$

and

$$w^{(n)} \succeq (s\beta\tau)^{(n)}.$$

(ii) Suppose a_j, a_k either both exceed n or both do not exceed n . For $k \leq d \leq n-1$, let $s_{c(d)}$ be the largest integer, $r < s_{c(d)} \leq n$, such that $s_{c(d)} \notin \{|a_1|, \dots, |a_d|\}$ (if no such integer exists, we let $s_{c(d)} = r$). Then

$$\begin{aligned} w^{(j)} &\succeq (a_1, \dots, a_{j-1}, a'_k) \uparrow, \\ w^{(d)} &\succeq (a_1, \dots, \hat{a}_j, \dots, \hat{a}_k, \dots, a_d, s'_{c(d)}, s') \uparrow, \quad k \leq d \leq n-1, \end{aligned}$$

and

$$w^{(n)} \succeq (s\beta\tau)^{(n)}.$$

(c) Let $\alpha = \varepsilon_j$, $1 \leq j \leq n$. For $j \leq d \leq n-1$, let $s_{m(d)}$ be the largest integer, $|a_j| < s_{m(d)} \leq n$, such that $s_{m(d)} \notin \{|a_1|, \dots, |a_d|\}$ (if no such $s_{m(d)}$ exists, we let $s_{m(d)} = |a_j|$). Then

$$w^{(d)} \succeq (a_1, \dots, \hat{a}_j, \dots, a_d, s'_{m(d)}) \uparrow, \quad j \leq d \leq n-1,$$

and

$$w^{(n)} \succeq (s\beta\tau)^{(n)}.$$

III. The special orthogonal group $\text{So}(2n)$. Let

$$E = \begin{pmatrix} 0 & & & 1 \\ & \ddots & & \\ & & \ddots & \\ 1 & & & 0 \end{pmatrix}_{2n \times 2n},$$

and let $(,)$ be the symmetric bilinear form on k^{2n} , represented by E , with respect to $\{e_1, \dots, e_{2n}\}$. Let

$$(19) \quad G = \text{So}(2n) = \{A \in \text{SL}(2n) \mid {}^t A E A = E\}.$$

Let σ be the involution on $\text{SL}(2n)$ defined by

$$(20) \quad \sigma(A) = E({}^t A)^{-1} E, \quad A \in \text{SL}(2n).$$

We have

$$(21) \quad \text{So}(2n) = \text{SL}(2n)^\sigma.$$

As in §§I and II, we obtain, in view of (21), identifications (described below) for W and W^{Pa} . We have

$$(22) \quad W = \left\{ (a_1 \cdots a_{2n}) \in S_{2n} \left| \begin{array}{l} (1) a_i = 2n + 1 - a_{2n+1-i}, \quad 1 \leq i \leq 2n, \\ (2) \#\{i, 1 \leq i \leq n \mid a_i > n\} \text{ is even} \end{array} \right. \right\}.$$

For $1 \leq d \leq n$, let

$$(23) \quad Z_d = \left\{ (a_1, \dots, a_d) \left| \begin{array}{l} (1) \quad 1 \leq a_1 < a_2 < \dots < a_d \leq 2n, \\ (2) \quad \text{for } 1 \leq i \leq 2n, \text{ if } i \in \{a_1, \dots, a_d\}, \text{ then} \\ \quad 2n + 1 - i \notin \{a_1, \dots, a_d\} \end{array} \right. \right\}.$$

We have for $d \neq n - 1$

$$(24) \quad W^{P_d} = Z_d.$$

For $d = n - 1$, if $w \in W^{P_d}$, then

$$(25) \quad w \equiv wu_i \pmod{W_{P_{n-1}}}, \quad 0 \leq i \leq n, \quad i \neq n - 1,$$

where

$$(26) \quad u_i = \begin{cases} s_{\alpha_n} & \text{if } i = n, \\ \text{Id} & \text{if } i = 0, \\ s_{\alpha_i} s_{\alpha_{i+1}} \cdots s_{\alpha_{n-2}} s_{\alpha_n} & \text{if } 1 \leq i \leq n - 2. \end{cases}$$

(Here Id denotes the identity element in W .) In particular, for $w_1, w_2 \in W$, say $w_1 = (a_1 \cdots a_{2n})$, $w_2 = (b_1 \cdots b_{2n})$, we can have $w_1^{(n-1)} = w_2^{(n-1)}$ without $(a_1, \dots, a_{n-1}) \uparrow$ and $(b_1, \dots, b_{n-1}) \uparrow$ being the same. Thus $W^{P_{n-1}}$ gets identified with a *proper* subset of Z_{n-1} (cf. Definition (23)). For $w \in W$, say $w = (c_1 \cdots c_{2n})$, we have

$$(27) \quad w^{(d)} = (c_1, \dots, c_d) \uparrow, \quad d \neq n - 1.$$

To describe $w^{(n-1)}$, we let, for $1 \leq i \leq n$, $i \neq n - 1$,

$$(28) \quad (y_1^{(i)}, \dots, y_{n-1}^{(i)}) = \begin{cases} \text{the } (n-1)\text{-tuple given by the first } (n-1) \\ \text{entries in } wu_i \end{cases}$$

and

$$(29) \quad Y = \{(y_1^{(i)}, \dots, y_{n-1}^{(i)}) \uparrow, 0 \leq i \leq n, i \neq n - 1\}.$$

We observe that Y is totally ordered under \geq (cf. (3)). We have

$$(30) \quad w^{(n-1)} = \text{the smallest (under } \geq \text{) element in } Y.$$

Unlike the cases of $\text{Sp}(2n)$ (resp. $\text{So}(2n + 1)$), the Bruhat order in W , the Weyl group of $\text{So}(2n)$, is not induced from the Bruhat order in S_{2n} . Hence the Bruhat order in W^{P_d} does not coincide with the natural order on d -tuples (cf. (3)). We now describe the Bruhat order in W^{P_d} .

For $1 \leq i \leq 2n$, let

$$i' = 2n + 1 - i \quad \text{and} \quad |i| = \min\{i, i'\}.$$

Under the above identification, given two elements $(a_1, \dots, a_d), (b_1, \dots, b_d)$ in W^{P_d} , $1 \leq d \leq n$, we have (cf. [10])

$$(a_1, \dots, a_d) \geq (b_1, \dots, b_d)$$

if and only if the following two conditions hold:

$$(A) \quad (a_1, \dots, a_d) \geq (b_1, \dots, b_d).$$

(B) Suppose for some r , $1 \leq r \leq d$, and some i , $0 \leq i \leq d - r$,

$$(|a_{i+1}|, \dots, |a_{i+r}|) \uparrow = (|b_{i+1}|, \dots, |b_{i+r}|) \uparrow = \{n + 1 - r, \dots, n\}.$$

Then

$$\#\{j, i + 1 \leq j \leq i + r \mid a_j > n\}$$

and

$$\#\{j, i + 1 \leq j \leq i + r \mid b_j > n\}$$

should both be odd or both even.

PROPOSITION D.1. (Assume $\text{char } k \neq 2, 3$.) Let $G = \text{So}(2n)$. Let $w, \tau \in W$, with $w \succeq \tau$, and let $\tau = (a_1 \cdots a_{2n})$. Then the tangent space $T(w, \tau)$ to $X(w)$ at $e(\tau)$ is spanned by the set of root vectors $\{X_{-\beta}, \beta \in N(w, \tau)\}$, where $N(w, \tau)$ is the subset of $\tau(R^+)$ consisting of roots β which satisfy criteria (a) and (b) below. Let $\beta = \tau(\alpha)$, $\alpha \in R^+$.

(a) Let $\alpha = \varepsilon_j - \varepsilon_k$, $1 \leq j < k \leq n$. Then

$$w \succeq s_\beta \tau.$$

(b) Let $\alpha = \varepsilon_j + \varepsilon_k$, $1 \leq j < k \leq n$. Let s (resp. r) be the minimum (resp. maximum) of $\{|a_j|, |a_k|\}$.

(i) Suppose precisely one of $\{a_j, a_k\}$ does not exceed n . Then

$$\begin{aligned} w^{(j)} &\succeq (a_1, \dots, a_{j-1}, a'_k) \uparrow, \\ w^{(k)} &\succeq (a_1, \dots, \hat{a}_j, \dots, a_{k-1}, r, s') \uparrow, \\ w^{(n-1)} &\succeq (s_\beta \tau)^{(n-1)}, \end{aligned}$$

and

$$w^{(n)} \succeq (s_\beta \tau)^{(n)}.$$

(ii) Suppose a_j, a_k either both exceed n or both do not exceed n . For $k \leq d \leq n - 2$, let $s_{-l(d)}, \dots, s_{-1}, s_0, s_1, \dots, s_{c(d)}$ be the integers

$$s < s_{-l(d)} < s_{-l(d)+1} < \cdots < s_{-1} < s_0 = r < s_1 < \cdots < s_{c(d)} \leq n$$

such that

$$s_i \notin \{|a_1|, \dots, |a_d|\}, \quad -l(d) \leq i \leq c(d), \quad i \neq 0.$$

Then

$$\begin{aligned} w^{(j)} &\succeq (a_1, \dots, a_{j-1}, a'_k) \uparrow, \\ w^{(d)} &\succeq \begin{cases} (a_1, \dots, \hat{a}_j, \dots, \hat{a}_k, \dots, a_d, s'_{c(d)-1}, s') \uparrow & \text{if } (l(d), c(d)) \neq (0, 0), \\ (a_1, \dots, \hat{a}_j, \dots, \hat{a}_k, \dots, a_d, r', s') \uparrow & \text{if } (l(d), c(d)) = (0, 0), \end{cases} \end{aligned}$$

and for $d = n - 1$ or n ,

$$w^{(d)} \succeq (s_\beta \tau)^{(d)}.$$

