

RESEARCH ANNOUNCEMENTS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 16, Number 2, April 1987

CR MAPPINGS OF FINITE MULTIPLICITY AND EXTENSION OF PROPER HOLOMORPHIC MAPPINGS

M. S. BAOUENDI, S. R. BELL AND LINDA PREISS ROTHSCHILD

1. Introduction. We shall describe some general theorems about CR mappings between three-dimensional manifolds which, among other results, imply that any proper holomorphic mapping $f: D \rightarrow D'$ between pseudoconvex domains in \mathbf{C}^2 with real analytic boundaries extends to be holomorphic in a neighborhood of the closure of D (Theorem 8). In case the domain D is strictly pseudoconvex, this result follows from the classical Lewy-Pinčuk reflection principle [9, 11]. In case D' is strictly pseudoconvex, or in case D and D' are given by polynomial defining functions, f extends by [2]. In case the proper mapping f is biholomorphic, the extendability has been proved by Baouendi, Jacobowitz, and Treves [1]. The general case of a proper holomorphic mapping between weakly pseudoconvex domains which is not biholomorphic is more complicated because branching might occur. We have developed a method in the spirit of [1] which allows us to prove extendability at boundary points even if branching occurs (Theorems 3 and 6).

The mapping $f(z, w) = (z^2, w)$ which maps the domain $\mathbf{E} = \{(z, w) \in \mathbf{C}^2: |z|^4 + |w|^2 < 1\}$ onto the unit ball in \mathbf{C}^2 has the property that it maps points of type four (in the sense of Kohn [8]) in the boundary of \mathbf{E} to points of type two in the boundary of the ball. Furthermore, the local branching order of f at these points is two. We prove that this phenomenon holds in general. If M and M' are abstract three-dimensional CR manifolds, and $H: M \rightarrow M'$ a CR mapping, there is a notion of multiplicity of H at $p_0 \in M$, for which the type of p_0 is equal to the multiplicity at p_0 times the type of $H(p_0)$. Theorems 1 and 2 state these results more precisely. Theorems 5 and 7 give applications of these results and of the extendability result (Theorem 3) to CR and proper self-mappings.

Received by the editors October 9, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32D15, 32H99.

First author supported by NSF Grant DMS 8603176; second author supported by NSF Grant DMS 8420754 and the Sloan Foundation; third author supported by NSF Grant DMS 8601260.

©1987 American Mathematical Society
0273-0979/87 \$1.00 + \$.25 per page

2. Main results and applications. A real smooth manifold M is called a *CR manifold* if there is a subbundle \mathcal{V} of \mathbf{CTM} , the complexified tangent bundle of M , satisfying the conditions $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, and $\mathcal{V} \cap \overline{\mathcal{V}} = (0)$; \mathcal{V} is called the *CR bundle* of M .

If M and M' are two CR manifolds with CR bundles \mathcal{V} and \mathcal{V}' , a CR mapping from M into M' is a smooth mapping $H: M \rightarrow M'$, such that at every $p \in M$

$$(1) \quad H'(\theta) \in \mathcal{V}'_{H(p)}, \quad \forall \theta \in \mathcal{V}_p,$$

where \mathcal{V}_p is the fiber of \mathcal{V} at p , $\mathcal{V}'_{H(p)}$ is the fiber of \mathcal{V}' at $H(p)$, and $H': \mathbf{CTM} \rightarrow \mathbf{CTM}'$ is the differential map of H .

From now on we restrict ourselves to the case: $\dim_{\mathbf{R}} M = \dim_{\mathbf{R}} M' = 3$, $\dim_{\mathbf{C}} \mathcal{V} = \dim_{\mathbf{C}} \mathcal{V}' = 1$. Let $p_0 \in M$, $p'_0 = H(p_0) \in M'$, and let L, L' be two nonvanishing smooth sections of \mathcal{V} and \mathcal{V}' , defined near p_0 and $H(p_0)$ respectively. It follows from (1) that there exists a smooth function λ defined in a neighborhood of p_0 in M such that, for all $p \in M$ near p_0 ,

$$(2) \quad H'(L_p) = \lambda(p)L'_{H(p)}.$$

DEFINITION. The CR mapping $H: M \rightarrow M'$ is of *finite multiplicity* at p_0 if there exists a differential operator of the form

$$(3) \quad S = M_1 \cdots M_j$$

(called *string* of length j) with $M_p = L$ or \overline{L} , such that

$$(4) \quad S\lambda(p_0) \neq 0.$$

The mapping is said to be of *multiplicity k* if the shortest string S for which (4) holds is of length $k - 1$.

In particular H is of multiplicity 1 if $\lambda(p_0) \neq 0$, of multiplicity 2 if $\lambda(p_0) = 0$, and either $L\lambda(p_0) \neq 0$ or $\overline{L}\lambda(p_0) \neq 0$, etc.

It is clear that for a CR mapping H to be of multiplicity k is independent of the choice of the vector fields L and L' .

Following Kohn [8], the CR manifold M is of *finite type* at p_0 if and only if any smooth complex vector field defined near p_0 is in the Lie algebra spanned by L and \overline{L} . It is of *type m* at p_0 ($m \geq 2$) if the shortest bracket of L 's and \overline{L} 's not in the span of L and \overline{L} at p_0 , is of length m . (We have used the following convention: the length of $[L, \overline{L}]$ is 2, that of $[L, [L, \overline{L}]]$ is 3, etc.)

We can now state our first result.

THEOREM 1. Let $H: M \rightarrow M'$ be a CR mapping from M into M' . If H is of multiplicity k at $p_0 \in M$ ($1 \leq k < \infty$), and M' is of type m' at $p'_0 = H(p_0)$ ($2 \leq m' < \infty$), then:

- (i) M is of type m at p_0 with $m = km'$.
- (ii) If S is a string of the form (3) with length $\leq k - 1$, and with at least one $M_p = \overline{L}$, then $S\lambda(p_0) = 0$; in particular $L^{k-1}\lambda(p_0) \neq 0$.
- (iii) $H'(\mathbf{CT}_{p_0}M) \not\subset \mathcal{V}'_{H(p_0)} \oplus \overline{\mathcal{V}}_{H(p_0)}$.

Note that if $H'_{p_0}: T_{p_0}M \rightarrow T_{H(p_0)}M'$ is the differential map at p_0 , then (iii) implies that

$$H'_{p_0} \neq \{0\}.$$

If $k = 1$, it follows from (2), (4) (with $S = 1$), and (iii) that H'_{p_0} is an isomorphism; therefore H is a local diffeomorphism from a neighborhood of p_0 onto a neighborhood of $H(p_0)$. This fact, when H is a boundary value of a holomorphic mapping from a pseudoconvex domain in \mathbb{C}^2 to another, follows from Derridj [5].

The proof of Theorem 1 uses recursive arguments with repeated applications of the following two observations.

First, suppose A and B are vector fields on M for which there exist smooth functions $\alpha_j(u)$, $\beta_j(u)$ on M , $1 \leq j \leq r$, and smooth vector fields A'_j and B'_j on M' such that

$$H'(A_u) = \sum_j \alpha_j(u)A'_{j,H(u)}, \quad H'(B_u) = \sum_j \beta_j(u)B'_{j,H(u)};$$

then

$$\begin{aligned} H'([A, B]_u) &= \sum_j (A\beta_j)(u)B'_{j,H(u)} - \sum_j (B\alpha_j)(u)A'_{j,H(u)} \\ &\quad + \sum_{p,q} (\alpha_p\beta_q)(u)[A'_p, B'_q]_{H(u)}. \end{aligned}$$

Second, if E_1 and E_2 are two commutators of L and \bar{L} of length n_1 and n_2 respectively then

$$[E_1, E_2] = aL + b\bar{L} + \sum_{\alpha} c_{\alpha}C^{\alpha},$$

where each C^{α} is a commutator of L and \bar{L} of length $|\alpha| < n_1 + n_2$, and a, b, c_{α} are smooth functions on M .

One of the crucial steps in the proof of Theorem 1 consists of proving that if S_1 and S_2 are two strings of the form (3) with length $k - 1$, having the same number of L 's, then

$$S_1\lambda(p_0) = S_2\lambda(p_0).$$

We say that the mapping $H: M \rightarrow M'$ is *flat* at p_0 if all partial derivatives of H of any order vanish at p_0 .

The following result is a consequence of Theorem 1.

THEOREM 2. *Let $H: M \rightarrow M'$ be a CR mapping and $p_0 \in M$. If M and M' are of finite type at p_0 and $p'_0 = H(p_0)$ respectively, then the following conditions are equivalent:*

- (a) H is of finite multiplicity at p_0 .
- (b) $H'_{p_0} \neq \{0\}$.
- (c) H is not flat at p_0 .

Examples show that the conclusions of Theorems 1 and 2 are no longer valid if the finite type conditions are dropped in the assumptions.

Our main analyticity result is the following.

THEOREM 3. *Let M, M' be two real analytic CR manifolds, H a smooth CR mapping from M into M' , and $p_0 \in M$. If H is of finite multiplicity at p_0*

and M' of finite type at $p'_0 = H(p_0)$, then H is real analytic in a neighborhood of p_0 .

The proof of Theorem 3 uses the general approach of [1]. Since M and M' are real analytic, they can be considered as embedded in \mathbf{C}^2 , where the variables are denoted by z, w . We assume that $p_0 = H(p_0) = 0$, and that M and M' are respectively given locally by

$$\text{Im } w = \varphi(z, \bar{z}, \text{Re } w), \quad \text{Im } w = \psi(z, \bar{z}, \text{Re } w),$$

with $\varphi(z, 0, \text{Re } w) = \psi(z, 0, \text{Re } w) = 0$. The mapping H is then locally given by a pair of CR functions (f, g) defined on M and satisfying

$$(5) \quad \frac{g - \bar{g}}{2i} = \psi \left(f, \bar{f}, \frac{g + \bar{g}}{2} \right).$$

As in [1], it suffices to show that f and g are real analytic with respect to $\text{Re } w$, uniformly in z .

We have here $\lambda = L\bar{f}$, where the function λ is as in (2). By Theorem 1 we have $L^k \bar{f}(0) \neq 0$ and $L^j \bar{f}(0) = 0, 0 \leq j < k$.

An important part of the proof consists of repeatedly applying L to (5). In addition to the arguments used in [1] (where $k = 1$), the following result in one complex variable is crucial.

LEMMA. Let a be a positive number and R the domain in \mathbf{C} defined by $|\xi| < a, 0 < \eta < a$, with $\zeta = \xi + i\eta$. Let u, v be two functions defined in \bar{R} and satisfying:

- (i) $u, v \in C^\infty(\bar{R})$ and u, v are holomorphic in R ,
- (ii) $h(\xi) = u(\xi)/v(\xi) \in C^\infty([-a, a])$,
- (iii) there exists a positive integer p and, for $0 \leq j \leq p - 1$, functions $a_j \in C^\infty(\bar{R})$, holomorphic in $R, a_j(0) = 0$, such that

$$(h(\xi))^p + a_{p-1}(\xi)(h(\xi))^{p-1} + \dots + a_0(\xi) = 0, \quad \text{in } [-a, a].$$

Then h extends holomorphically to R as $u(\zeta)/v(\zeta)$, and $u/v \in C^\infty(R \cup (-a, a))$.

For real analytic CR mappings, we have the following result, which gives a justification for the definition of finite multiplicity.

THEOREM 4. Let M, M' be two real analytic CR manifolds of finite type at p_0 and p'_0 respectively, M connected, and $H: M \rightarrow M'$ a real analytic CR mapping with $H(p_0) = p'_0$.

(i) If H is not of finite multiplicity at p_0 , then H is constant, i.e. $H(M) = p'_0$.

(ii) If H is of multiplicity k at p_0 ($1 \leq k < \infty$), then p'_0 is an interior point of $H(M)$. More precisely, for every U , a sufficiently small neighborhood of p_0 in M , there exists V , an open neighborhood of p'_0 in M' such that $V \subset H(U)$. In addition there is a finite number of real analytic curves $\gamma_1, \dots, \gamma_r$ contained in V such that, for every $p' \in V \setminus (\bigcup_{j=1}^r \gamma_j \cup \{p'_0\})$, there exist exactly k points

$p_1, \dots, p_k \in U$ satisfying $H(p_j) = p'$, $1 \leq j \leq k$, with H of multiplicity 1 at each p_j .

Theorems 3 and 4 together with an argument involving the iteration of H , and the use of properties of real analytic sets, yield the following global result.

THEOREM 5. *Let M be a real analytic compact CR manifold and $H: M \rightarrow M$ a smooth CR mapping. If M is of finite type at each point and H of finite multiplicity at each point of M , then H is of multiplicity one at each point. Therefore H is a local analytic diffeomorphism.*

Several of the previous results have applications to holomorphic extendability of proper maps in domains in \mathbf{C}^2 . Indeed it is well known that the boundary value of a holomorphic mapping is a CR mapping, and, when the boundaries are real analytic, the question of holomorphic extendability of a CR mapping reduces to that of its real analyticity. We give here some of these applications.

THEOREM 6. *Let D and D' be two open bounded sets of \mathbf{C}^2 with real analytic boundaries. Let $F: D \rightarrow D'$ be a proper holomorphic mapping with $F \in C^\infty(\bar{D})$. If F is nowhere flat on ∂D then F extends holomorphically to a neighborhood of \bar{D} . More precisely there exist D_1 and D'_1 , two open bounded neighborhoods of \bar{D} and \bar{D}' respectively, with real analytic boundaries, such that F extends as a holomorphic proper mapping from D_1 into D'_1 .*

Using an argument due to Pinčuk [10] and Theorems 5 and 6 we obtain the following, which generalizes a result of Bedford and Bell [3].

THEOREM 7. *Let D be an open bounded set in \mathbf{C}^2 with real analytic boundary and F a proper holomorphic self-mapping of D . If $F \in C^\infty(\bar{D})$ and F is nowhere flat on ∂D then F extends as a biholomorphism from an open neighborhood of \bar{D} onto another.*

In the pseudoconvex case, using a generalized form of an argument due to Fornaess [7] the nowhere flatness can be dropped in Theorem 6. Then making use of Theorem 6 and the result of Bell and Catlin [4], and Diederich and Fornaess [6], we obtain

THEOREM 8. *Let D and D' be two bounded pseudoconvex domains in \mathbf{C}^2 with real analytic boundaries. If F is a proper holomorphic map from D into D' then F extends as a proper holomorphic mapping from a neighborhood of \bar{D} to a neighborhood of \bar{D}' .*

Complete proofs will be published elsewhere.

REFERENCES

1. M. S. Baouendi, H. Jacobowitz and F. Trèves, *On the analyticity of CR mappings*, Ann. of Math. **122** (1985), 365–400.
2. E. Bedford and S. Bell, *Extension of proper holomorphic mappings past the boundary*, Manuscripta Math. **50** (1985), 1–10.
3. ———, *Proper self maps of weakly pseudoconvex domains*, Math. Ann. **261** (1982), 47–49.

