this the reader will have to consult other works (such as Stein-Weiss [**S-W**], Stein [**S**], and the more recent Garcia-Cuerva–Rubio de Francia [**G-R**]).

This is a book for mature readers. The author does not hesitate to use ideas and results from diverse branches of mathematics—special functions, functional analysis, partial differential equations, differential geometry. But for the reader with a strong background, or a willingness to accept a non-self-contained presentation, this book offers many pleasures. In addition to the concrete computational results already mentioned, the book contains concise and insightful presentations of a number of important abstract topics, including induced representations, representations of compact groups, conformal transformations, Clifford algebras and spinors. Taylor has an original point of view, and is able to bring new insights to familiar topics.

REFERENCES

[**F-S**] G. Folland and E. M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 429–522.

[**G-R**] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland, 1985.

[**H**] R. Howe, *On the role of the Heisenberg group in harmonic analysis*, Bull. Amer. Math. Soc. (N.S.) **3** (1980), 821–843.

[**S**] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970.

[**S-W**] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, N. J., 1971.

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Near-rings and their links with groups, by J. D. P. Meldrum, Pitman Publishing, Boston, London, Melbourne, 275 pp., $44.95. ISBN 0-470-20648-9.

Many mathematicians do not highly appreciate theories which have prefixes like near-, semi-, hemi-, para-, quasi-, and so on. This certainly should not apply in the case of near-rings. As the name suggests, a *near-ring* is a "generalized ring"; more precisely, the commutativity of addition is not required and just one of the distributive laws is postulated.

Near-rings arise very naturally in the study of mappings on groups. If $(G, +)$ is a group (not necessarily abelian) then the set $M(G)$ of all mappings from G to G is a near-ring with respect to pointwise addition and composition of mappings. If G is abelian and if one only takes "linear" maps

(= endomorphisms of G), one gets the well-known endomorphism ring End G. From that, one already sees that the generalization from rings to near-rings basically is the transition from linearity to nonlinearity. And this is a matter truly worth being considered.

At the beginning of this century, L. E. Dickson **[6]** constructed the first examples of "proper" near-rings. Actually, these near-rings were *near-fields* (= near-rings in which the nonzero elements form a multiplicative group). Dickson started with a field $F = (F, +, \cdot)$, took a "*coupling map*" Φ: $F - \{0\} \to \mathrm{Aut}(F, +)$, $f \to \phi_f$ with $\phi_f \circ \phi_g = \phi_{(f\phi_g)\cdot g}$ (we write xh for the image of x under the function h). Defining $f \cdot_\Phi g := (g\phi_f) \cdot g$ and $0 \cdot_\Phi f := 0$,

$$F^\Phi := (F, +, \cdot_\Phi),$$

the "Φ-*derivation*" of F, is a near-field, and in general not a field. Near-fields which arise as F^Φ from some field F are now called *Dickson near-fields*. Hence Dickson showed that the second distributive law does not follow from the remaining axioms of a skew-field. But the commutativity of addition does, and this is certainly not trivial.

THEOREM 1 (B. H. NEUMANN **[12]**, KARZEL **[10]**, ZEMMER **[19]**, ET AL.) *The additive group of a near-field is abelian.*

In 1936, H. Zassenhaus succeeded in finding all finite near-fields:

THEOREM 2 (ZASSENHAUS **[18]**). *A finite near-field is either a Dickson near-field, or it belongs to* 8 *exceptional near-fields or orders* 2, 5^2, 7^2, 11^2 (*two near-fields*), 23^2, 29^2 *or* 59^2.

The 7 exceptional near-fields of order p^2 in Theorem 2 can easily be described by means of 2×2-matrices. The exceptional near-field of order 2 is $(\mathbf{Z}_2, +\, *)$ with $x * y = y$. For more details see, e.g., **[3]**.

From 1907 on, Veblen, and Wedderburn **[15]** and many other authors used near-fields N in order to coordinatize geometric planes $\mathcal{G}$, so the points of $\mathcal{G}$ are just the elements of $N \times N$ and the lines of $\mathcal{G}$ are given by all $\{(x, xa + b) \mid x \in N\}$ and $\{(c, x) \mid x \in N\}$ for a, b, $c \in N$, $a \neq 0$. For more information on this topic see e.g. **[8]** or **[5]**.

Another application of near-fields is in the description of all *sharply 2-transitive* permutation groups G on some set X. This means that for all distinct x_1, $x_2 \in X$ and all distinct y_1, $y_2 \in X$ there is precisely one $g \in G$ with $x_i g = y_i$ $(i = 1, 2)$.

THEOREM 3 (ZASSENHAUS **[18]**, BLACKBURN AND HUPPERT **[3]**). *For every finite sharply 2-transitive permutation group G there is a finite near-field F such that G is isomorphic to all transformations* $x \to ax + b$ $(a, b \in F,\ a \neq 0)$ *from F to F.*

Thus, by Theorem 3 in conjunction with Theorem 2, all finite sharply 2-transitive groups can be considered to be "known." More can be found in **[3]** and [**11**]. Near-fields have numerous other applications, mostly to geometry, which are described in the new book [**16**].

J. D. P. Meldrum's book does give some information on these near-fields, but it concentrates much more on the exciting (and mostly quite recent) theories and applications of "proper" near-rings (which are neither near-fields nor rings). It consists of two parts. Part 1 gives a very readable and thoughtful introduction to the subject, mentions many examples and some applications, and then proceeds quickly to the structure theory of near-rings (radicals, semisimplicity, primitive and simple near-rings). The main difference from the existing book [**13**] is that Meldrum's book is much easier to read and concentrates on the core of the theory, while [**13**] was in some way intended to be encyclopedic.

At a first glance, the structure theory (initiated in [**17**] and in the dissertation [**4**], supervised by E. Artin) of near-rings seems to run along very similar lines as the one of ring theory. It does, to a large extent, in the results; but most of the deeper results need completely different proofs and approaches. As an example, let me present Theorem 3.35 in the book, which many people consider to be the basic structure result on near-rings.

If N is a near-ring and if $(G, +)$ is a group such that for $n \in N$ and $g \in G$ a "product" gn is defined such that $g(n_1 + n_2) = gn_1 + gn_2$, $(gn_1)n_2 = g(n_1 n_2)$ hold for all $g \in G$ and $n_1, n_2 \in G$, then G is called an *N-group* (or *N-near-module*). Every group G is, for example, an $M(G)$-group and also a $\mathbf{Z}$-group in the obvious ways. For any near-ring N, $N_0 := \{n \in N \mid 0n = 0\}$ is called its *zero-symmetric part.* N_0 is a subnear-ring of N. If $N = M(G)$, $N_0 =: M_0(G)$ consists of the maps $G \to G$ which map 0 into 0. If G is an N-group with $GN \neq 0$ and without proper subgroups H such that $HN_0 \subseteq H$, G is called *irreducible*. If the *annihilator* $A(G_N) := \{n \in N \mid Gn = 0\}$ is 0, G is said to be a *faithful* N-group. A near-ring N with a faithful, irreducible N-group G is called *primitive* (on G). Since for near-rings it makes sense to study different concepts of primitivity, the described concept is more precisely called "2-primitivity" in the book. An endomorphism $h \in \operatorname{End} G$ is called an *N-homomorphism* if $h(gn) = h(g)n$ for all $g \in G$, $n \in N$. $\operatorname{Hom}_N(G, G)$ is the collection of all N-homomorphisms of the N-group G.

If S is a set of endomorphisms of a group $(G, +)$,

$$M_S(G) := \{f \in M_0(G) \mid f \circ s = s \circ f \text{ for all } s \in S\}$$

is an important subnear-ring of $M(G)$. If N and M are subnear-rings of $M(G)$ with $N \subseteq M$, we say that N is *dense* in M if for all $k \in \mathbf{N}$, all distinct $g_1, \ldots, g_k \in G$ and all $m \in M$ there is some $n \in N$ with $g_i n = g_i m$ ($i = 1, \ldots, k$). If we take, for $g \in G$ and $m \in M$, all $\{m' \in M \mid gm' = gm\}$ as a base of a topology, the resulting concept of density is the one just defined.

We are now ready for the following result; parts of it were first obtained by H. Wielandt (unpublished manuscripts). The form as given now is due to G. Betsch [**1**].

DENSITY THEOREM. *Let $N = N_0$ be a near-ring with identity which is primitive on G.*

(i) *If N is a ring then Jacobson's density theorem* [**9**] *applies: N is a dense subring of* $\mathrm{Hom}_F(G, G)$, *where F is the skew-field* $\mathrm{Hom}_N(G, G)$.

(ii) *If N is not a ring then N is a dense subnear-ring of $M_S(G)$, where* $S := \mathrm{Aut}(G, +) \cap \mathrm{Hom}_N(G, G)$ *is a fixed-point-free automorphism group on G.*

The proof is certainly nontrivial. Part (ii) has to be set apart from case (i) by a careful investigation of the lattice of "ideals" of the N-group G. Many variants, generalizations, specializations, and applications of this theorem are known. A convenient special case is

COROLLARY. *Let the "non-ring" $N = N_0$ with identity be primitive on G, such that S (as in the Density Theorem) $= \{\mathrm{id}\}$. Then N is dense in $M_0(G)$.*

Of course, for finite G (more generally for near-rings N with a suitable descending chain condition), "density" is the same as "equality." Some applications of the Density Theorem are mentioned in [**14**].

Part 2 of Meldrum's book considerably differs from the first part. It starts from the observation that the sum of two endomorphisms of a nonabelian group $(G, +)$ is, in general, not an endomorphism any more. Hence it makes sense to study the collection of all sums/differences of endomorphisms of G, i.e., the additive closure $E(G)$ of $\mathrm{End}\, G$ in $M(G)$. It quickly turns out that $E(G)$ is a near-ring (subnear-ring of $M(G)$); $E(G)$ is the most prominent example of a "*distributively generated*" near-ring (that is, a near-ring which is additively generated by "distributive elements"). These distributively generated near-rings are thoroughly investigated in part 2 of J. D. P. Meldrum's book. After presenting the basic facts of distributively generated near-rings (in particular: of *endomorphism near-rings* $E(G)$), the author studies their structure, considers free distributively generated near-rings and "group distributively generated near-rings", thus initiating a "nonlinear representation theory" for groups. A similar sounding, but different, topic is how to represent an arbitrary distributively generated near-ring as a subnear-ring of a suitable $E(G)$. Since the relation between G and $E(G)$ is, of course, much stronger than that between G and $M(G)$, a lot of group theory enters part 2 of the book. The information flow is by no means just one-sided from groups to endomorphism near-rings, but goes the other way, too: near-ring theory faithfully pays back to groups (part of) the amount groups have invested into near-rings. For example, the following result can be derived from the Density Theorem mentioned above (J. D. P. Meldrum gives a very nice new proof of it):

THEOREM 4. *For a group G, every map $G \to G$ fixing 0 is the sum/difference of inner automorphisms iff G is a finite, simple, nonabelian group.*

There are, of course, many interesting open problems in this area. A "nonabelian homological algebra," initiated by A. Fröhlich in [**7**], is still to be developed. A striking problem, withstanding all attacks up to now, is the following.