BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 17, Number 2, October 1987

THE STRUCTURE OF ALGEBRAIC THREEFOLDS: AN INTRODUCTION TO MORI'S PROGRAM

JÁNOS KOLLÁR

CONTENTS

Introduction. This article intends to present an elementary introduction to the emerging structure theory of higher-dimensional algebraic varieties. Introduction is probably not the right word; it is rather like a travel brochure describing the beauties of a long cruise, but neglecting to mention that the first half of the trip must be spent toiling in the stokehold. Perusal of brochures might give some compensation for lack of royal roads.

Having this limited aim in mind, the prerequisities were kept very low. As a general rule, geometry is emphasized over algebra. Thus, for instance, nothing is used from abstract algebra. This had to be compensated by using more results from topology and complex variables than is customary in introductory algebraic geometry texts. Still, aside from some harder results used in occasional examples, only basic notions and theorems are required.

Throughout the history of algebraic geometry the emphasis constantly shifted between the algebraic and the geometric sides. The first major step was a detailed study of algebraic curves by Riemann. He approached the subject from geometry and analysis, and gave a quite satisfactory structure theory. Subsequently the German school, headed by Max Noether, introduced algebra to the subject and problems arising from algebraic geometry substantially influenced the development of commutative algebra, especially the works of Emmy Noether and Krull.

During the same period the Italian school of Castelnuovo, Enriques, and Severi investigated the geometry of algebraic surfaces and achieved a satisfactory structure theory. Their work, however, lacked the Hilbertian rigor, and

Received by the editors December 3, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14-02, 14E30, 14E35, 32J25, 14J10, 14J15, 14E05, 14J30.

©1987 American Mathematical Society
0273-0979/87 $1.00 + $.25 per page

therefore started to fall into disrepute. When the students tried their hands at studying algebraic threefolds, their conclusions were frequently false and none of the proofs of the deep results stood up to even their own standards.

Several attempts were made to place algebraic geometry on solid foundations. After substantial achievements of van der Waerden, this was accomplished by Weil and Zariski through a systematic use of commutative algebra. A good indication of this style is the two-volume treatise, *Commutative algebra*, by Zariski and Samuel, which grew out of an attempt to write an introductory chapter to a planned algebraic geometry book.

The approach via commutative algebra was further developed by Nagata, and it culminated in the unfinished magnum opus, *Eléments de géométrie algébrique* by Grothendieck. In his treatment, commutative algebra and algebraic number theory became special cases of algebraic geometry. A spectacular success of this point of view is Faltings' proof of the Mordell conjecture.

By the end of the sixties, the foundational work was mostly done and attention turned toward more classical problems. First the theory of curves and surfaces was redone and completed. A structure theory of threefolds seemed, however, intractable.

In 1972 Iitaka proposed some bold and interesting conjectures concerning higher-dimensional varieties. Along that path Ueno produced in 1977 the first structure theorem about threefolds. It was clear, however, that the scope of their approach was limited. One major stumbling block was the lack of a good analog of the so-called minimal models of surfaces.

The major breakthrough came in 1980. Mori introduced several new ideas and accomplished the first major step toward proving the existence of minimal models. At the same time Reid defined and investigated minimal models (assuming their existence) and pointed out various ways of using them. Since then algebraic threefolds have been investigated intensively and recently these efforts resulted in the proof of several deep structure theorems.

The aim of this article is to give an accessible outline of these results.

Chapters 2–5 provide a short introduction to algebraic geometry. Chapter 6 discusses the topology of algebraic varieties from the point of view of Mori's theory. Classically, one attempted to study a variety by studying its codimension-one subvarieties. A fundamental observation of Mori's theory is that one should investigate the one-dimensional subvarieties as well. This turns out to be related to some simple topological properties of algebraic varieties. After some further introductory material, Chapter 8 gives a general discussion of the basic strategy and aim of the structure theory. Chapter 9 outlines how most of this program can be accomplished for surfaces. Chapter 10 is devoted to Mori's seminal paper. These results are sufficient to complete the structure theory of surfaces, but they provide only the first step in general. In dimension three, Mori's program leads to the study of certain singular varieties. His results are reworked in this more general context in Chapter 11. The last remaining step, the Flip Theorem, is discussed in Chapter 12, and the last chapter is devoted to further results.

Since this article is aimed at a general readership, I saw no point in giving references to technical research papers. Instead I provide a short annotated bibliography at the end which should satisfy the needs of most readers.

Part of this article was written while I enjoyed the hospitality of École Polytechnique. Financial support was also provided by the National Science Foundation (DMS-8503734).

A preliminary version of this article was circulated in the fall of 1986. Several mistakes, inaccuracies and misprints were called to my attention by R. Friedman, L. Lempert, K. Matsuki, T. Oda, M. Reid, H. Rossi, Y. Shimizu, and G. Tian. H. Clemens and S. Mori pointed out several conceptual problems in the presentation which led to substantial revisions. I am very grateful for all their attention and help.

2. What is algebraic geometry?

2.1. STARTING POINT. After Descartes introduced coordinates on the plane, it became clear that simple and much studied geometric objects (e.g., lines and conics) can be defined by simple polynomial equations (linear, resp. quadratic). This suggests that as a next step one could try to study curves defined by higher-degree polynomials. Already Newton studied plane cubics in depth. Two problems, however, tended to make results cumbersome. The first one is the missing infinity. Two different lines mostly intersect in a point, but sometimes they are parallel. It turns out to be very convenient to claim that they intersect at infinity. This leads to the introduction of the projective plane $\mathbf{RP}^2$. The other problem is apparent already with one-variable polynomials: the roots of a decent-looking polynomial can be lurking in the complex plane far away from the reals. Even when the real picture seems good, the explanation of certain phenomena might be seen only by studying them over $\mathbf{C}$: For example, why does the Taylor series of $(1 + x^2)^{-1}$ refuse to converge everywhere? Therefore we replace $\mathbf{R}$ by $\mathbf{C}$ and get $\mathbf{CP}^2$. There is no reason to stop with dimension 2 and so we introduce:

2.2. DEFINITION of $\mathbf{CP}^n$. As a point set this consists of $(n + 1)$-tuples $(x_0 : \dots : x_n)$ such that $(x_0 : \dots : x_n)$ and $(\lambda x_0 : \dots : \lambda x_n)$ are identified for $0 \neq \lambda \in \mathbf{C}$. We exclude $(0 : \dots : 0)$.

Let $U_i = \{(x_0 : \dots : x_n) \in \mathbf{CP}^n \mid x_i \neq 0\}$. The map $\phi_i \colon \mathbf{C}^n \to \mathbf{CP}^n$ given by $(y_1, \dots, y_n) \to (y_1 : \dots : y_i : 1 : y_{i+1} : \dots : y_n)$ maps $\mathbf{C}^n$ onto U_i in a 1:1 way. Since $(0 : \dots : 0)$ is not in $\mathbf{CP}^n$, the U_i cover it, and using the usual topology of $\mathbf{C}^n$, the maps ϕ_i put a topology on $\mathbf{CP}^n$. It is easy to see that $\mathbf{CP}^n$ becomes a real $2n$-dimensional compact manifold. Despite this we shall count complex dimensions and say that $\mathbf{C}^n$ and $\mathbf{CP}^n$ are n-dimensional. The usual notion of dimension will be referred to as real or topological dimension. This is twice the complex dimension.

2.3. DEFINITION OF ALGEBRAIC VARIETIES. We want to consider subsets of $\mathbf{CP}^n$ that can be defined by polynomial equations. An arbitrary polynomial $f(x_0, \dots, x_n)$ would not do, because $f(x_0, \dots, x_n)$ and $f(\lambda x_0, \dots, \lambda x_n)$ need not be both zero. However, if f is homogeneous of degree m, then $f(\lambda x_0, \dots, \lambda x_n) = \lambda^m f(x_0, \dots, x_n)$ and the set of zeroes of f in $\mathbf{CP}^n$ makes sense. Therefore we say that a subset $V \subset \mathbf{CP}^n$ is a projective algebraic variety if there are homogeneous polynomials $f_1, \dots, f_k$ such that $V = \{(\mathbf{x}) \in \mathbf{CP}^n \mid f_1(\mathbf{x}) = \cdots = f_k(\mathbf{x}) = 0\}$. One can easily see that the intersection or union of two projective algebraic varieties is again projective algebraic.

The difference of two projective varieties is not projective usually; these are called quasiprojective varieties.

Of special interest are the varieties of the form $V \cap U_i$ where V is projective. With the notation of 2.2, if V is defined by equations $f_1(\mathbf{x}) = \cdots = f_k(\mathbf{x}) = 0$ then $\phi_i^{-1}(V) \subset \mathbf{C}^n$ is defined by equations $f_j(y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n) = 0$, $j = 1, \ldots, k$. Such subsets of $\mathbf{C}^n$ are called affine varieties. The projective variety V can be thought of as being patched together from the affine varieties $V \cap U_i$.

2.4. DEFINITION. Algebraic geometry is the branch of mathematics that studies algebraic varieties.

2.5. CONCEPTUAL PROBLEMS. It is not clear that it is reasonable to study algebraic varieties. The forced union of algebra and geometry might be a bad match. There are three basic questions that have to be answered satisfactorily in order to believe that we are after something interesting.

(i) What is the relationship between the variety and its equations? Completely different equations might define the same variety!

(ii) Why is the definition so global? Maybe we should work locally on $\mathbf{CP}^n$!

(iii) Are we doing intrinsic or extrinsic geometry? Why do we rely so much on the ambient space $\mathbf{CP}^n$?

The rest of this chapter will be devoted to answering these questions.

Answer to Problem (i)

2.6. It is easier to study the affine case, i.e., varieties in $\mathbf{C}^n$. As an illustration we consider the case of plane curves. Let $H = \{(x, y) \in \mathbf{C}^2 \mid h(x, y) = 0\}$, $F = \{(x, y) \in \mathbf{C}^2 \mid f(x, y) = 0\}$. We are interested in deciding when $H \subset F$.

Let $h = \prod h_i$ be its decomposition into irreducible factors and $H_i = \{(x, y) \in \mathbf{C}^2 \mid h_i(x, y) = 0\}$. Clearly $H = \cup H_i$ and thus we have to study the conditions $H_i \subset F$. Let g be one of the h_i's, and G the corresponding curve.

2.7. PROPOSITION. *Let $g, f \in \mathbf{C}[x, y]$ and assume that g is irreducible. Let G, F be the corresponding curves. Then $G \subset F$ iff g divides f.*

PROOF. In general this requires a little field theory, but the case $g = x^2 + y^2 - 1$ gives a good illustration. First suppose that f has rational coefficients and pick any transcendental number, say e. Pick e' such that $g(e, e') = 0$, i.e., $e' = \sqrt{1 - e^2}$. I claim that $f(e, e') = 0$ iff g divides f. We divide f by g to get

$$f(x, y) = (y^2 + x^2 - 1)v(x, y) + yp(x) + q(x).$$

If $f(e, e') = 0$ then $e'p(e) + q(e) = 0$. Substituting $e' = \sqrt{1 - e^2}$ and getting rid of the square root gives

$$(1 - e^2)p^2(e) = q^2(e).$$

Since f has rational coefficients so does p and q, hence the above gives a polynomial identity for e, which is impossible unless the two sides are equal:

$$(1 - x^2)p^2(x) = q^2(x).$$

This is impossible since $1 - x^2$ is not a square, so p and q are identically zero. That is what we wanted.