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Tilings and patterns, by Branko Grünbaum and G. C. Shephard. W. H. Freeman and Company, New York, 1987, ix + 700 pp., $59.95. ISBN 0-7176-1193-1.

From time immemorial artisans and artists have constructed ingenious tilings and ornaments using repeated motives. This is demonstrated in the introduction of the beautiful volume under review by numerous examples from widely separated cultures. However, the importance of tilings and patterns in crystallography and some related branches of science was recognized only towards the end of the last century. From this time on many crystallographers, chemists, physicists, architects, engineers, and mathematicians have been working in this field. Although they accumulated a vast literature in books and periodicals, "much effort has been wasted duplicating previously known results." When the authors started collecting material for this book, they were surprised to find "how little about tilings and patterns is known," and how many errors were made because of "badly formulated definitions and lack of rigor."

For more than a decade the authors were busy critically revising the earlier results and making significant contributions to the theory of tilings and ornaments in a series of papers of their own. Their effort is crowned by the unique comprehensive monograph *Tilings and patterns*, which lays a solid foundation for one of the most attractive fields in geometry.

The book gives evidence of the sound didactic sense of the authors. The introduction of new concepts is carefully prepared, often supported by convincing intuitive arguments, and most formal definitions are richly illustrated by figures or some other means. The exposition is informal, but always clear and exact. Most sections contain carefully selected exercises, which often assign the completion of some arguments in the text to the reader, and many promising research problems. The "Notes and References" at the end of the chapters point out many interesting connections with other disciplines in mathematics, art, and science, and give an exhaustive survey of the related results. Consequently, the book turns out to be much more than "just" a fundamental scientific work. It is also an excellent textbook, which is meant not only for students and professional mathematicians, but for anyone interested, by profession or hobby, in geometrical configurations. The intriguing topic, the beautiful figures, and the profusion of the challenging open problems will be an inexhaustible source of pleasure and inspiration for generations to come. In addition, the book will certainly contribute to the restoration of the balance between abstract and intuitive trends in our antigeometrical age.

The bulk of the material presented in the book originates from the last few decades, and a great part of it has never been published before. Because of the wealth of this material we cannot but throw a cursory glance at some parts of each chapter.

From the "Basic Notions" of Chapter 1 let us mention the less well known concept of a k-isohedral tiling. This is a tiling whose tiles form exactly k so-called transitivity classes, each class consisting of tiles equivalent under the symmetry group of the tiling. The terms k-isogonal and k-isotoxal tiling are used in the same sense, except that now the vertex set and the edge set, respectively, fall into k transitivity classes.

Scrutinizing the illustrations of various notions we get acquainted with surprising tilings, e.g., Voderberg's spiral tiling, and analogues of crystals having "fault lines" of other singularities. We were surprised to learn, among many intriguing facts about tilings, how difficult it is to decide whether the plane can be tiled with congruent copies of a given polygon.

A great part of Chapter 2 offers itself for being introduced in the high school curriculum. Even pupils with literary interests would enjoy the beautifully presented enumeration of the Archimedean tilings and their coloring. As a natural generalization of the Archimedean tilings, which are 1-isogonal (in short isogonal), k-isogonal tilings with regular tiles are also discussed. Following Islamic traditions and Kepler's pioneering investigations, especially interesting new tilings are presented which contain starshaped tiles. The principle of duality is demonstrated by the isohedral tilings with regular vertex-figures which are connected with the name of Laves.

Postponing the discussion of some kinds of "pathological" tilings, Chapter 3 deals with "well-behaved" tilings. The authors impose a series of restrictions on the tilings. This enables them to phrase and prove important properties of the respective families of tilings. The literature contains many errors due to the lack of such restrictions. One important restriction is the "normality" of a tiling T, which means that (i) every tile of T is a topological disc, (ii) the intersection of any two tiles of T is either empty or a connected set, and (iii) the inradii of the tiles have a uniform positive lower bound and the circumradii of the tiles have a uniform finite upper bound. After pointing out some consequences of normality, further restrictions are made in order to prove "Euler's Theorem for Tilings" and some relations between the tilings defined by these restrictions. We quote only one of the many attractive unsolved problems contained in this chapter. Suppose that each tile of a normal monohedral tiling has at least one boundary point in common with exactly n other tiles. It is then conjectured that $n \leqslant 21$; but "no finite upper bound for n has ever been established."

After introducing the notions of topological and combinatorial equivalence of tilings in Chapter 4 it is proved that for normal tilings the two notions are the same. A third kind of equivalence is the following: two tilings are said to be isotopic if there is a family of transformations that continuously deforms one into the other. It is characteristic of the whole exposition that this notion is vivified by referring to Escher's famous woodcut *Metamorphosis* III. The authors conjecture that isotopy coincides with topological equivalences. Although they point out the difficulty inherent in the problem, it is surprising to learn that in spite of our present highly developed knowledge in topology and combinatorics such a challenging question has not been settled yet.

An interesting part of Chapter 4 investigates the relations between dual tilings. Probably for the first time in the literature, the authors emphasize the limit of the applicability of the idea of duality to tilings. The next sections yield a thorough, in part new, treatment of the topologically vertex-, tile-, and edge-transitive (homeogonal, homeohedral, homeotoxal) normal tilings and their classification, correcting deficiencies in earlier literature and pointing out marked differences between normal and non-normal tilings.

Chapter 5 contains the first mathematical theory of patterns. Starting with the everyday meaning of the word "pattern" as a repetition of a "motif" in a "regular" manner, the authors seem to have endeavored to define a mathematically manageable class of patterns, called discrete patterns, by imposing as weak conditions as possible. They propose a natural classification of discrete patterns which is based on the symmetry group of the pattern and some of its subgroups. As a preparation for the actual enumeration of the possible pattern types in the next chapter, various relationships between patterns and tilings are established.

Chapter 6 contains one of the main results presented in the book: the first satisfactory classification and the complete enumeration of all types of isohedral, isogonal, and isotoxal tilings. As a matter of fact, the classification is carried out for "marked" tilings, i.e., tilings each of whose tiles (vertices, or edges) carries a "motif". Such a tiling can be considered as a pattern. The classification according to pattern types is refined by introducing "incidence symbols" which describe the relation of a tile (vertex, or edge) to its neighborhood. These symbols provide not only a principle of classification but enable us to actually enumerate all possible types of tilings with the corresponding transitivity properties.

Chapter 7 presents a basic concept which underlies the classification of all kinds of geometrical "objects" (systems of sets such as topological discs, lines, etc.). Intuitively speaking, two objects are said to be homeomeric if there is a homeomorphism which maps one object into the other so that all symmetries of both objects remain preserved. It is shown that most of the pattern and tiling types defined in the previous chapters coincide with the respective homeomeric types. However, for patterns whose motifs are open topological