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ON PROBLEMS OF U. SIMON CONCERNING MINIMAL SUBMANIFOLDS OF THE NEARLY KAEHLER 6-SPHERE

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ABSTRACT. We classify the complete 3-dimensional totally real submanifolds with sectional curvature $K \geq \frac{1}{16}$ in the nearly Kaehler 6-sphere $S^6(1)$, and, as a corollary, we solve a problem for compact 3-dimensional totally real submanifolds of $S^6(1)$ related to U. Simon's conjecture for compact minimal surfaces in spheres.

1. The nearly Kaehler 6-sphere. It is well known that a 6-dimensional sphere S^6 does not admit any Kaehler structure, and whether S^6 does or does not admit a complex structure, as far as we know, is still an open question. However, using the Cayley algebra \mathcal{E} , a natural *almost complex structure* J can be defined on S^6 considered as a hypersurface in \mathbf{R}^7 , which itself is viewed as the set \mathcal{E}_+ of the purely imaginary Cayley numbers (see, for instance, E. Calabi [1]). Together with the *standard metric* g on S^6 , J determines a *nearly Kaehler structure* in the sense of A. Gray [9], i.e. one has $\forall X \in \mathcal{L}(S^6): (\tilde{\nabla}_X J)(X) = 0$, where $\tilde{\nabla}$ is the Levi Civita connection of g . For reasons of normalization only, in the following we will always work with this nearly Kaehler structure on the sphere $S^6(1)$, of radius 1 and constant sectional curvature 1. The compact simple Lie group G_2 is the group of automorphisms of \mathcal{E} and acts transitively on $S^6(1)$. Moreover, G_2 preserves both J and g .

2. Special submanifolds of $(S^6(1), g, J)$. With respect to J , two natural particular types of submanifolds M of $S^6(1)$ can be investigated: those which are *almost complex* (i.e. for which the tangent space of M at each point is invariant under the action of J) and those which are *totally real* (i.e. for

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which the tangent space of M at each point is mapped into the normal space by J). There only exist 2-dimensional almost complex submanifolds in $S^6(1)$, and these are always minimal [10]. Curvature properties for such surfaces were first obtained by K. Sekigawa [15]. Totally real submanifolds of $S^6(1)$ have either dimension 2 or 3. N. Ejiri [7] showed that every 3-dimensional totally real submanifold of $S^6(1)$ is orientable and minimal, and he first obtained curvature properties for such submanifolds. The 3-dimensional totally real submanifolds of $S^6(1)$ were also considered by J. B. Lawson Jr. and R. Harvey [11] in their study of calibrated geometries and by K. Mashimo [13] in his classification of such compact submanifolds which are orbits of closed subgroups of G_2 . In our study of submanifolds of the nearly Kaehler 6-sphere, we concentrated on the following problems.

PROBLEM A. Which real numbers can be realised as the constant sectional curvatures of almost complex or minimal totally real submanifolds M of $S^6(1)$?

PROBLEM B. Let K_1 and K_2 be two consecutive numbers as in Problem A. Then, do there exist compact submanifolds M of $S^6(1)$ whose sectional curvatures K satisfy $K_1 \leq K \leq K_2$, other than those for which $K \equiv K_1$ or $K \equiv K_2$?

3. On minimal submanifolds of arbitrary spheres. For minimal surfaces in a unit sphere $S^n(1)$ of arbitrary dimension n , one has a complete answer to Problem A (given by O. Boruvka, E. Calabi and N. Wallach for the case of positive Gauss curvature, the solutions being $K = 2/m(m+1)$, $m \in \mathbf{N}_0$, and by R. Bryant, proving the nonexistence of minimal surfaces of constant negative Gauss curvature in any sphere). Concerning Problem B, U. Simon conjectured the following.

CONJECTURE OF U. SIMON [12]. *Let M be a compact surface whose Gauss curvature K satisfies $2/m(m+1) \leq K \leq 2/m(m-1)$, for some $m \in \mathbf{N} \setminus \{0, 1\}$, which is minimally immersed in $S^n(1)$. Then $K \equiv 2/m(m+1)$ or $K \equiv 2/m(m-1)$ (and hence M is a Boruvka sphere).*

For $m = 2$ and $m = 3$, this conjecture is known to be true, as was shown by H. B. Lawson Jr., U. Simon, M. Kothe, K.-D. Semmler, K. Benko and M. Kozłowski. Recently, quite a number of people have been working on this problem; in particular, T. Ogata, S. Montiel, T. Itoh, G. Jensen, M. Rigoli, J. Bolton, L. Woodward and U. Simon, A. Schwenk, B. Opozda together with the present authors. As far as we know however, in general this conjecture is still open for $m \geq 3$. In view of U. Simon's conjecture, we would like to call problems of type A and B, as stated above for almost complex and totally real submanifolds of $S^6(1)$, "*problems of U. Simon*".

4. Solutions of problems A and B.

THEOREM 1 [15]. *If an almost complex surface M in $S^6(1)$ has constant Gauss curvature K , then either $K = 1$ (and M is totally geodesic) or $K = \frac{1}{6}$ or $K = 0$.*

THEOREM 2 [4, 2]. *Let M be a compact almost complex surface in $S^6(1)$ with Gauss curvature K .*

- (i) Let $\frac{1}{6} \leq K (\leq 1)$; then either $K \equiv \frac{1}{6}$ or $K \equiv 1$.
- (ii) If $0 \leq K \leq \frac{1}{6}$, then either $K \equiv 0$ or $K \equiv \frac{1}{6}$.

THEOREM 3 [6]. *If a minimal totally real surface M in $S^6(1)$ has constant Gauss curvature K , then either $K = 1$ (and M is totally geodesic) or $K = 0$.*

THEOREM 4 [6]. *For a compact minimal totally real surface M in $S^6(1)$ with nonnegative Gauss curvature K (or equivalently $0 \leq K \leq 1$), either $K \equiv 0$ or $K \equiv 1$.*

In 1981, making use of a special choice of local orthonormal frames, N. Ejiri solved Problem A in the remaining case as follows.

THEOREM 5 [7]. *If a 3-dimensional totally real submanifold M of $S^6(1)$ has constant sectional curvature K , then either $K = 1$ (and M is totally geodesic) or $K = \frac{1}{16}$.*

Totally real 3-dimensional totally geodesic submanifolds in $S^6(1)$ are not hard to construct. On the other hand, N. Ejiri [8] proved that $S^3(\frac{1}{16})$ can be immersed totally real and isometrically in $S^6(1)$. K. Mashimo [13] found an orbit of a closed subgroup of G_2 with constant curvature $\frac{1}{16}$. Later we will explicitly describe these immersions, obtaining for instance as extra information that they are in fact 56-fold coverings of $S^3(\frac{1}{16})$. Compared to the solutions given in Theorems 2 and 4, the solution of Problem B seems more involved in the present case. In our approach, the solution represented by Theorem 6 is an immediate consequence of the Main Theorem.

THEOREM 6. *A compact 3-dimensional totally real submanifold of $S^6(1)$ whose sectional curvature K satisfies $\frac{1}{16} \leq K \leq 1$ has constant sectional curvature $K = \frac{1}{16}$ or $K = 1$.*

MAIN THEOREM. *Let $x: M^3 \rightarrow S^6(1)$ be a totally real isometric immersion of a complete 3-dimensional Riemannian manifold M^3 into the nearly Kaehler 6-sphere $S^6(1)$. If the sectional curvatures K of M^3 satisfy $K \geq \frac{1}{16}$, then either M^3 is simply connected and x is G_2 -congruent to $x_1: M_1 \rightarrow S^6(1)$ (in which case K attains all values in the closed interval $[\frac{1}{16}, \frac{21}{16}]$) or to $x_2: M_2 \rightarrow S^6(1)$ (in which case $K \equiv 1$), or else \tilde{x} , the composition of the universal covering map of M^3 with x , is G_2 -congruent to $x_3: M_3 \rightarrow S^6(1)$ (in which case $K \equiv \frac{1}{16}$).*

SKETCH OF PROOF (details will appear elsewhere [3]). As in our partial solution of Problem B [5], a crucial role is played by some integral formulas of A. Ros, of which we'll state one next. We do believe that these formulas provide a powerful tool to study problems in global Riemannian geometry.

LEMMA OF A. ROS [14]. *Let M be a compact Riemannian manifold, UM its unit tangent bundle and UM_p the fibre of UM over a point p of M . Let dp , du and du_p denote the canonical measures on M , UM and UM_p , respectively. For any continuous function $f: UM \rightarrow \mathbf{R}$, one has*

$$\int_{UM} f \, du = \int_M \left\{ \int_{UM_p} f \, du_p \right\} dp.$$

Now, let T be any k -covariant tensor field on M . Then one has the integral formula $\int_{UM} (\nabla T)(u, u, \dots, u) du = 0$, where ∇ is the Levi Civita connection on M .

We apply this lemma for some particular tensors T constructed in terms of the second fundamental form h of the immersion x . Then, under the assumption $K \geq \frac{1}{16}$, amongst others, we obtain that

$$R(v, A_{J_v}v; A_{J_v}v, v) = \frac{1}{16} \{ \|A_{J_v}v\|^2 - \langle A_{J_v}v, v \rangle^2 \}$$

for all $p \in M^3$ and all $v \in UM_p^3$, where R is the Riemann-Christoffel curvature tensor of M^3 and A is the Weingarten map with respect to a normal section ξ . From this, working with special frames, using the Gauss equation and with the help of computer manipulation of formulas, we can prove that at each point p the second fundamental form h_p has either one of three possible forms, leading respectively to the possibilities $K(p) \equiv 1$, $K(p) \equiv \frac{1}{16}$ and $K(p) \in [\frac{1}{16}, \frac{21}{16}]$, where $K(p)$ is the sectional curvature function of M^3 at p . In the following, we will give comments concerning only x_1 (x_2 is the totally geodesic case, and for x_3 we will confine ourselves to give precise formulas for the immersion). The existence of x_1 is guaranteed by the following result taken from a preprint by N. Ejiri.

THEOREM OF N. EJIRI [8]. *Let M be a 3-dimensional simply connected Riemannian manifold with metric $\langle \cdot, \cdot \rangle$. Suppose there exist a $(1, 2)$ -symmetric tensor field T on M such that*

- (i) $\text{Tr } T = 0, \langle T(X, Y), Z \rangle = \langle T(X, Z), Y \rangle,$
- (ii) $\langle R(X, Y)W, Z \rangle = \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle + \langle T(X, Z), T(Y, W) \rangle - \langle T(X, W), T(Y, Z) \rangle,$
- (iii) $(\nabla_X T)(Y, Z) - (\nabla_Y T)(X, Z) + T(Z, X \wedge Y) = 0$, where \wedge is the vector product determined by some orientation on M .

Then, up to a transformation of G_2 , there exists a unique isometric immersion x of M into S^6 as a totally real submanifold with second fundamental form $J(x_*T)$ and with normal connection D defined by $D_X J(x_*Y) = J(x_*(\nabla_X Y + X \wedge Y))$.

Namely, on the unit sphere $S^3(1) = \{y = (y_1, y_2, y_3, y_4) \in \mathbf{R}^4 \mid \sum y_j^2 = 1\}$ we can define a metrix $\langle \cdot, \cdot \rangle$, vector product \wedge and tensor field T satisfying the conditions of this theorem and for which K attains all values in $[\frac{1}{16}, \frac{21}{16}]$. This leads to the immersion $x_1: S^3(1) \subset \mathbf{R}^4 \rightarrow S^6(1) \subset \mathbf{R}^7: y \rightarrow z = (z_1, \dots, z_7)$, where

$$\begin{aligned} z_1(y) &= \frac{1}{9}(5y_1^2 + 5y_2^2 - 5y_3^2 - 5y_4^2 + 4y_1), \\ z_2(y) &= -\frac{2}{3}y_2, \quad z_3(y) = \frac{2\sqrt{5}}{9}(y_1^2 + y_2^2 - y_3^2 - y_4^2 - y_1), \\ z_4(y) &= \frac{\sqrt{3}}{9\sqrt{2}}(-10y_1y_3 - 2y_3 - 10y_2y_4), \quad z_5(y) = \frac{\sqrt{15}}{9\sqrt{2}}(2y_1y_4 - 2y_4 - 2y_2y_3), \\ z_6(y) &= \frac{\sqrt{15}}{9\sqrt{2}}(2y_1y_3 - 2y_3 + 2y_2y_4), \quad z_7(y) = -\frac{\sqrt{3}}{9\sqrt{2}}(10y_1y_4 + 2y_4 - 10y_2y_3). \end{aligned}$$

In practice x_1 was found solving the system of differential equations (1) on p. 67 of M. Spivak's volume IV [16]; the rigidity of course follows from the fundamental theorem of submanifolds.

Finally, we mention the formulas of $x_3: S^3(\frac{1}{16}) = \{y \in \mathbf{R}^4 \mid \sum y_j^2 = 16\} \subset \mathbf{R}^4 \rightarrow S^6(1) \subset \mathbf{R}^7: y \mapsto z(y)$; we have

$$z_1(y) = \sqrt{15} \cdot 2^{-10} \cdot (y_1y_3 + y_2y_4) \cdot (y_1y_4 - y_2y_3)(y_1^2 + y_2^2 - y_3^2 - y_4^2),$$

$$z_2(y) = 2^{-12} \left[-\sum_j y_j^6 + 5 \sum_{i < j} y_i^2 y_j^2 (y_i^2 + y_j^2) - 30 \sum_{i < j < k} y_i^2 y_j^2 y_k^2 \right],$$

$$z_3(y) = 2^{-10} [y_3y_4(y_3^2 - y_4^2)(y_3^2 + y_4^2 - 5y_1^2 - 5y_2^2) + y_1y_2(y_1^2 - y_2^2)(y_1^2 + y_2^2 - 5y_3^2 - 5y_4^2)],$$

$$z_4(y) = 2^{-12} \{y_2y_4(y_2^4 + 3y_3^2 - y_4^4 - 3y_1^4) + y_1y_3(y_3^4 + 3y_2^2 - y_1^4 - 3y_4^4) + 2(y_1y_3 - y_2y_4)[y_1^2(y_2^2 + 4y_4^2) - y_3^2(y_4^2 + 4y_2^2)]\},$$

$$z_5(y_1, y_2, y_3, y_4) = z_4(y_2, -y_1, y_3, y_4),$$

$$z_6(y) = \sqrt{6} \cdot 2^{-12} \cdot [y_1y_3(y_1^4 + 5y_2^2 - y_3^4 - 5y_4^4) - y_2y_4(y_2^4 + 5y_1^2 - y_4^4 - 5y_3^4) + 10(y_1y_3 - y_2y_4)(y_3^2y_4^2 - y_1^2y_2^2)],$$

$$z_7(y_1, y_2, y_3, y_4) = z_6(y_2, -y_1, y_3, y_4).$$

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