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Groups and geometric analysis. Integral geometry, invariant differential operators and spherical functions, by Sigurdur Helgason. Academic Press, Orlando, 1984, xviii + 654 pp., $39.50. ISBN 0-12-338301-3

This is a book on harmonic analysis, but harmonic analysis is like the proverbial elephant: it looks very different to different people. To some it means maximal theorems and BMO; to others it means parametrizing the unitary dual; for Harish-Chandra it was the Plancherel Theorem for semisimple groups. Thus no book on harmonic analysis will be universal: each can present only a small part of the subject. To define the book at hand it will help to consider it briefly in its relation to harmonic analysis as a whole, and to examine the types of problems it addresses and does not address.

For purposes of this book, harmonic analysis on a homogeneous space X of a Lie group G means answering the following questions (see p. 2). Write $X = G/K$ for an appropriate closed subgroup K of G.

First, find the algebra $\mathbf{D}(G/K)$ of G-invariant differential operators on X. Then

(B) Describe the spaces of joint eigenfunctions for the operators in $\mathbf{D}(G/K)$;

(A) Decompose "arbitrary" functions on $X = G/K$ into (superpositions of) joint eigenfunctions of $\mathbf{D}(G/K)$; and

(C) Determine on which joint eigenspaces the natural action of G, by translation of functions, is irreducible.

Not all of the book actually fits this formulation, but much of it does. How does a program based on these problems relate to harmonic analysis as a whole?

We first observe that A, B, C imply a decidedly "concrete" stance toward harmonic analysis, as opposed to an "abstract" one. Classification of representations is not a question here. We are dealing with function spaces of a very concrete sort rather than disembodied locally compact spaces, or even, say, sections of vector bundles.

Second, even within very concrete harmonic analysis, one need not restrict one's attention to homogeneous spaces. Fascinating results about a Hamiltonian action of a torus on a symplectic manifold have been discovered recently **[GS1, GS2]**, but these results are trivial if the action is transitive. And the action of a classical group on several copies of its defining vector space (e.g., O_n on $(\mathbf{R}^n)^m$) is the context for a surprisingly rich theory **[Ge, Ho]**.

Third, given that we will work on a homogeneous space, we observe, as does the author, that the formulation A, B, C puts some conditions on G and X: the algebra $\mathbf{D}(G/K)$ must be large enough to be interesting but small enough to be abelian. One much-studied situation **[FS, Gd, HN]**, in which there are no invariant differential operators is a semidirect product

$$X = (\mathbf{R}^{+\times} \times N)/\mathbf{R}^{+\times} \simeq N$$

where

(a) N is a connected simply connected nilpotent Lie group,

(b) $\mathbf{R}^{+\times}$ acts on N by "dilations"—a one parameter family of expansive automorphisms. The simplest example is to take $N = \mathbf{R}^n$, and to let $\mathbf{R}^{+\times}$ act by the standard scalar dilations,

$$a(x_1, \ldots, x_n) = (ax_1, ax_2, \ldots, ax_n), \qquad x_i \in \mathbf{R},\ a \in \mathbf{R}^{+\times}.$$

In fact, this example occurs in the book (theory of Riesz potentials, pp. 131–139), but the role of dilations is suppressed. Contrariwise, there are many situations where $\mathbf{D}(G/K)$ is nonabelian, e.g., complex flag varieties $X = X/T$ where K is a compact Lie group and $T \subseteq K$ is a maximal torus (see Exercise 14, p. 486).

However, the class of examples for which problems A, B, C make a sensible program is still very rich. Most prominent among them are the symmetric spaces of various sorts: Euclidean, compact, Riemannian, and "pseudo-Riemannian" or "affine." The author's earlier, now classic book (*Differential geometry and symmetric spaces*, Academic Press, 1962, updated as *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, 1978) treated the basic geometry of these spaces. The book under review is primarily concerned with aspects of harmonic analysis on them. Even within this more focussed area, choices of topic are necessary. In particular the pseudo-Riemannian symmetric spaces, which have been actively studied only relatively recently and whose theory is still developing, are mentioned very little in the book.

Having located the book's subject matter, we now describe it briefly.

There is a lot of meat here, much of it of prime quality. Each chapter is cut thick (some are almost books in themselves). Rather than attempt a slice-by-slice description, I will just try to convey the general flavor. The sub-title, *Integral geometry, invariant differential operators and spherical functions* goes far towards summarizing the book's concerns. Each of Integral Geometry and Spherical Functions has a long chapter devoted to it (Chapters I and IV respectively). More will be said below concerning the basics of spherical functions. The chapter on integral geometry is largely concerned with geometrically defined integral transforms, of which a prominent example is the Radon transform. This is a mapping from functions on $\mathbf{R}^n$ to functions on the space of hyperplanes in $\mathbf{R}^n$, which has the structure of a line bundle over $\mathbf{RP}^{n-1}$. Its definition is attractively naiive. Given a reasonable function f on $\mathbf{R}^n$, and a hyperplane ξ, define the Radon transform $\hat{f}(\xi)$ by

$$\hat{f}(\xi) = \int_{\xi} f(x)\, dm(x)$$

where $dm(x)$ is the standard Euclidean measure on ξ. An important problem about such transforms is to invert them. For the Radon transform, this can be done by observing that it is in a fairly straightforward way a factor of Fourier transform, and then using Fourier Inversion; or by constructing an adjoint map $^\vee$ from functions on hyperplanes to functions on $\mathbf{R}^n$ and then observing the composite $(\hat{f})^\vee$ is a (perhaps half-integral) negative power of the Laplacian (Theorem 2.13, p. 110).

But the dominant theme of the book is invariant differential operators. Their influence is everywhere, except perhaps Chapter V. We have already seen them in the formulation of problems A, B, and C. Chapter II is devoted entirely to them, and Chapter III, on invariant theory, contains among other things a thorough account of Chevalley's results on Weyl group invariants, which applies directly to the structure of invariant differential operators on symmetric spaces. They are used in Chapter II to invert integral transforms, and in Chapter IV to define spherical functions.

Besides the three subtitles, another important theme is integral transforms. In Chapter II these are geometrically defined, while Chapter IV studies the "Spherical Transform," defined by integrating against spherical functions, which gives the spectral decomposition of K-bi-invariant functions with respect to the invariant differential operators on a symmetric space. Major questions addressed concerning integral transforms include these three:

(i) Inversion Formulas: How can one recover a function from its transform?

(ii) Plancherel Theorems: How does the transform treat natural L^2 norms?

(iii) Paley-Wiener Theorems: How does the transform treat smooth functions of compact support?

A high point is the Paley-Wiener Theorem for the Spherical Transform, to which the author made significant contributions.

Chapter V, about compact groups and spaces, is somewhat separate from the rest of the book. §2 discusses lacunary Fourier Series, random Fourier Series and related topics in the context of a general compact (Lie) group. I found it illuminating.

There are many attractive and significant topics in this book, and many nice formulas. The author takes pains to give careful and usually complete proofs. There are a detailed table of contents, a glossary of notation, and historical notes. These features should make the book an excellent reference for the material it treats. (The index is not very complete, though.)

With such a book, it would be nice to go the whole hog, and tell everyone to buy it not only for themselves, but get their students to buy it too. Unfortunately, in its overall structure, the book has vagaries and lacunae in presentation with which I would not want a student of mine to have to struggle. Hence I cannot recommend it as a text. I also have some objections about content, but as issues of content are harder to debate, I will leave them aside. Not everyone may object to the features of the book that bother me. So what I will do is present one example of what I do not like in the book. It is not the only one, but it is probably the most serious. If this example does not bother you, probably my other objections won't either.

Although spherical functions are a major topic of the book, it does not provide the reader with a compact, conceptual overview of their many-sided nature. This failure has two aspects: first there is no overall summary of the many roles played by spherical functions, and how they are related; and second and worse, there is no explanation of the simple group-theoretic structure that binds all the different aspects together. The book should contain an amplified, but still quite compact, version of the following discussion.

Let G be a Lie group, $K \subseteq G$ a compact subgroup. Let us suppose, with the author, that $\mathbf{D}(G/K)$ is commutative. Spherical functions for the pair (G, K) have at least 6 aspects:

(i) They are K-bi-invariant eigenfunctions for the differential operators in $\mathbf{D}(G/K)$. More loosely: they are the solutions to certain eigenvalue problems.

(ii) They define homomorphisms of the convolution algebra of K-bi-invariant functions.

(iii) They define projections to eigenspaces of $\mathbf{D}(G/K)$.

(iv) They occur as proportionality factors in mean value formulas for eigenfunctions of $\mathbf{D}(G/K)$.

(v) They are matrix coefficients of representations of G with K-fixed vectors, and can be used to construct such representations.

(vi) They possess a certain "reproducing kernel" property.

To understand spherical functions, it is important first to realize these many different roles of spherical functions, and second to understand why the spherical functions can play them all, or better, why these are all just different points of view on the same thing.

The structure linking all these features is provided by the convolution algebra $\mathscr{E}'(G)$ of compactly-supported, K-bi-invariant distributions. (Here and below, we follow the notation of the book: $\mathscr{D}$ denotes compactly supported smooth functions, $\mathscr{E}$ arbitrary smooth functions, $\mathscr{D}'$ arbitrary distributions, and $\mathscr{E}'$ compactly-supported distributions; and indicates K-bi-invariance.)

The first point is to observe that $\mathscr{E}'(G)$ contains as subalgebras the algebra $\mathbf{D}(G/K)$ of G-invariant differential operators on G/K on one hand, and the algebra $\mathscr{D}(G)$ of K-bi-invariant smooth functions on the other. Thus properties (i) and (ii) are united in the fact that spherical functions define characters of $\mathscr{E}'(G)$. The algebra $\mathscr{E}'(G)$ is discussed along side $\mathbf{D}(G/K)$ in §5.1 of Chaper II, but the point that one may be regarded as a subalgebra of the other is not made. (If it were Corollary 5.4 would be simply an *a fortori* consequence of Corollary 5.2.) This is despite the fact that the universal enveloping algebra (case $K = \{1\}$) of G is identified to distributions supported at the identity.

Once one is aware of $\mathscr{E}'(G)$, property (vi) may be seen as another expression of the fact that spherical functions define characters of this algebra. (One convolves with the unit mass on a K double coset. This connection is implicit in the proof of Lemma 3.2, Chapter IV, p. 408.) And property (v) follows by examining how $\mathscr{E}'(G)$ acts on a K-fixed vector in a smooth representation of G.