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HARMONIC MEASURE IN CONVEX DOMAINS

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Introduction. Let Ω be an open, convex subset of $\mathbf{R}^N$. At almost every point x of $\partial\Omega$, with respect to surface measure $d\sigma$, there is a unique outer unit normal θ. The map $g\colon \partial\Omega \to S^n$ given by $g(x) = \theta$ and defined almost everywhere is called the Gauss map. (S^n, $n = N-1$, is the unit sphere in $\mathbf{R}^N$.) Suppose that the origin 0 belongs to Ω. Harmonic measure for Ω at 0 is the probability measure ω such that for all continuous functions f on $\partial\Omega$[1],

$$u(0) = \int_{\partial\Omega} f\, d\omega$$

where u solves the Dirichlet problem: $\Delta u = 0$ in Ω and $u = f$ on $\partial\Omega$.

Since Ω is a Lipschitz domain, Dahlberg's theorem **[4]** implies that $d\omega$ and $d\sigma$ are mutually absolutely continuous. Thus we can define a measure μ on S^n by $\mu = g_*\omega$ or

$$\mu(E) = \omega(g^{-1}(E)) \quad \text{for all } E \subseteq S^n.$$

We would like to pose the inverse problem: Given a probability measure μ on S^n, is there a domain Ω for which $\mu(E) = \omega(g^{-1}(E))$? Loosely speaking, we would like to find the convex domain given harmonic measure as a function of the unit normal.

We will solve the problem in case $d\mu = R\, d\theta$, R smooth and positive.

THEOREM 1. *For k an integer $\geq k(N)$ and $0 < \alpha < 1$, let $R \in C^{k,\alpha}(S^{N-1})$ be a positive function with $\int R\, d\theta = 1$. There exists a strictly convex domain Ω containing the origin, with $C^{k+2,\alpha}$ boundary, such that for $E \subset S^{N-1}$,*

$$\omega(g^{-1}(E)) = \int_E R\, d\theta,$$

where g is the Gauss map and ω is harmonic measure for Ω at 0. The domain Ω is unique up to dilation.

Our problem is natural for three reasons. First, it is analogous and closely related to the classical Minkowski problem. Second it is entirely solved in the plane by a continuous version of the well-known Schwarz-Christoffel formula. Third, the proof requires new, optimal estimates for

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[1]If Ω is unbounded, we suppose further that f tends to zero at infinity ($f \in C_0(\partial\Omega)$) and that u tends to zero at infinity.

harmonic measure in convex domains. These estimates dovetail with recent optimal estimates for the Monge-Ampère equation due to L. Caffarelli. The proof of Theorem 1 depends by way of Caffarelli's methods on the entire development of regularity theory in the Minkowski problem. We would like to thank L. Caffarelli for explaining his recent results and for his encouragement.

The Minkowski problem is to find a convex body Ω given Gaussian curvature K as a function of the unit normal. (See [3].) But K is the Jacobian determinant of the Gauss map; informally, $d\theta = K\,d\sigma$ or $d\sigma = (1/K)\,d\theta$. Thus the Minkowski problem can be rephrased: Find the convex domain Ω given surface measure $d\sigma$ as a function of the unit normal. In our problem we have simply replaced surface measure with harmonic measure.

The Schwarz-Christoffel formula

$$\Phi'(z) = re^{i\psi}\prod_{k=1}^{m}(z-x_k)^{-\beta_k}, x_1 < x_2 < \cdots < x_m, \qquad r > 0,\ \psi \in \mathbf{R},$$

gives the conformal mapping Φ from the upper half-plane $H = \{z \in \mathbf{C}\colon \operatorname{Im} z \geq 0\}$ to a polygon Ω with vertices $\Phi(x_1), \ldots, \Phi(x_m)$, $\Phi(\infty)$ and exterior angles $\pi\beta_1, \ldots, \pi\beta_m$, $2\pi - \sum_{k=1}^{m}\pi\beta_k$. We confine ourselves to the convex case: $0 < \beta_k < 1$ and $\sum_{k=1}^{m}\beta_k < 2$. (If $\sum_{k=1}^{m}\beta_k \leq 1$, then the polygon is unbounded, $\Phi(\infty) = \infty$.) Observe that $\arg\Phi'(x) = x - \sum_{k=1}^{m}\pi\beta_j$ for $x \in (x_{k-1}, x_k) k = 1, \ldots, m+1$, with the convention $x_0 = -\infty$, $x_{m+1} = \infty$. It follows that the outer normal to the side $\Phi((x_{k-1}, x_k))$ is $\alpha_k = x - \pi/2 - \pi\sum_{j=k}^{m}\beta_j$. So far Ω is only defined up to translation. We can fix Φ by $\Phi(i) = 0$. Let $c_k = \omega(\Phi((x_{k-1}, x_k)))$. Then $\mu = \sum c_k\delta_{\alpha_k}$. Furthermore, since the harmonic measure for Ω at 0 is the push-forward of the harmonic measure for H at i, $c_k = \frac{1}{\pi}\int_{x_{k-1}}^{x_k} dt/(1+t^2)$. Thus we can realize any μ that is a finite sum of delta functions by a suitable polygon Ω.

In general, for an arbitrary probability measure μ on $[0, 2\pi)$ define the monotone function $U\colon \mathbf{R} \to [0, 2\pi)$ by

$$U(x) = \min\left\{\alpha\colon \mu([0,\alpha]) \geq \frac{1}{\pi}\int_{-\infty}^{x}\frac{dt}{1+t^2}\right\}.$$

For any $E \subset [0, 2\pi)$, $\mu(E) = \frac{1}{\pi}\int_{U^{-1}(E)} dt/(1+t^2)$. Define a conformal mapping Φ of H by $\arg\Phi'(x) - \pi/2 = U(x)$. Thus

$$\Phi'(z) = \exp[-V(z) + i(U(z) + \pi/2)]$$

where $V(z)$ is the harmonic conjugate of U. If we normalize Φ by $\Phi(i) = 0$, then $g_*(\omega) = \mu$, as desired. Notice that V is unique up to an additive constant, so that Ω is unique up to dilation. Regularity properties of the Poisson integral and Hilbert transform imply that if $d\mu = R\,d\theta$ with $R \in C^{k,\alpha}(S^1)$, $k = 0, 1, 2, \ldots$ $0 < \alpha < 1$ and $R > 0$, then U and V belong to $C^{k+1,\alpha}$ and $\partial\Omega$ is a $C^{k+2,\alpha}$, strictly convex curve.

Estimates for harmonic measure. The estimates for harmonic measure that we need are expressed in terms of cross-sections of Ω. Let $H = \{x \in \mathbf{R}^N : (x - x_0) \cdot a \geq 0\}$ be a half-space. Let $E = \Omega \cap \partial H$ and $F = (\partial\Omega) \cap H$. Let $\frac{1}{2}E$ be the subset of E given by dilation by the factor $1/2$ from an origin defined as the center of mass of E. Let $\frac{1}{2}F$ be the set of points of F whose orthogonal projection onto ∂H lies in $\frac{1}{2}E$.

THEOREM 2. *Let Ω be a convex, open set in $\mathbf{R}^N$ such that $B_r \subset \Omega \subset B_1$. ($B_r$ is the ball of radius r about 0.) There is a constant C depending only on N and r such that*

(a) $\omega(F) \leq C\omega(\frac{1}{2}F)$,
(b) $\max_{x\in F} h(x) \leq C\omega(F)/\sigma(F)$, *where* $h = d\omega/d\sigma$.

These estimates should be compared with the case of Lipschitz domains. (See, e.g. [5].) Estimate (b) permits h to vanish but not to blow up. The key difference with the Lipschitz case is that instead of looking at "surface balls" of the form $B_t(x) \cap \partial\Omega$ we need to use the sets F, which resemble ellipses with uncontrolled eccentricity.

THEOREM 3. *Suppose that Ω is a convex domain $B_r \subset \Omega \subset B_1$, $\partial\Omega$ is C^1 and strictly convex. For any $\delta > 0$ there is $\varepsilon > 0$ depending only on δ, N, r, the C^1 modulus of continuity and the modulus of strict convexity such that*

$$\max_{x\in F} h \leq (1+\delta)\omega(F)/\sigma(F) \quad \textit{whenever } \sigma(F) \leq \varepsilon.$$

This improvement of (b) is analogous to the improvement of Dahlberg's theorem in [6].

The main ingredient in the proof of Theorem 2 beyond what is already known for Lipschitz domains is that Green's function is almost concave in the following sense:

LEMMA. *Let G be Green's function for Ω with pole at 0. ($\Delta G = -\delta$, so that $G \geq 0$.) There is a constant $C = C(N, r)$ such that*

$$G(x) + G(y) \leq CG((x+y)/2) \quad \textit{for all } x, y \in \Omega \setminus B_r.$$

Sketch of the Proof of Theorem 1. We find Ω by the method of continuity. Let $\tau_n = \int_{S^n} d\theta$. Let $R^t = (1-t)\tau_n^{-1} + tR$. We wish to show that $T = \{t \in [0,1]$: there is a strictly convex, $C^{k+2,\alpha}$ domain Ω^t with $d\mu^t = R_t\, d\theta\}$ is both open and closed in $[0,1]$. Since, $\Omega^0 = B_1$ yields $d\mu^0 = \tau_n^{-1}\, d\theta$, we have $0 \in T$, and it follows that $T = [0,1]$.

To show that T is open, we express the equation in terms of the Minkowski support function $u(\theta) = x \cdot \theta$ ($x = g^{-1}(\theta)$) on the sphere S^n. The domain Ω can be recovered from u by considering $F(r,\theta) = ru(\theta)$, a function in polar coordinates on $\mathbf{R}^N \cdot \nabla F$ is homogeneous of degree 0 and $\nabla F : S^n \to \partial\Omega$ is the inverse mapping of g. From now on we will regard functions and measures on S^n and $\partial\Omega$ as identified via g and ∇F. If $u_{ij} = \nabla_{ij} u$ denote covariant derivatives of u in an orthonormal frame on S^n, the Minkowski equation is (see [3])

$$\det(u_{ij} + u\delta_{ij}) = \frac{1}{K} = \frac{d\sigma}{d\theta}.$$

Since $h = d\omega/d\sigma$ and $R = d\omega/d\theta$, we have

$$(*) \qquad h\det(u_{ij} + u\delta_{ij}) = R.$$

The linearized equation is

$$\begin{aligned} Lv &\equiv \left. \frac{d}{ds} h_{u+sv} \det((u+sv)_{ij} + (u+sv)\delta_{ij}) \right|_{s=0} \\ &= (hc_{ij}v_i)_j - \frac{1}{K}\Lambda(hv), \end{aligned}$$

where $c_{ij}(u_{jk}+u\delta_{jk}) = (1/K)\delta_{ik}$ defines c_{ij} and Λ is the Neumann operator, that is the operator taking a function on $\partial\Omega$ to the normal derivative of its harmonic extension.

$\int R\,d\theta = 1$ implies $\int Lv\,d\theta = 0$ for all v and hence $L^*1 = 0$. By the theory of linear elliptic equations, the range of L is the orthogonal complement of the null space of L^*. (This is the only place at which the requirement $k \geq k(N)$ plays a role.) Thus, by the implicit function theorem, in order to prove that T is open we need only show that $L^*v = 0$ implies v is constant.

We will simultaneously find the null space of L and L^*. Notice that R does not change when u is replaced by $u + su$ since the corresponding region is the dilate $(1+s)\Omega$. Therefore, $Lu = 0$. Conversely, we have

PROPOSITION 1. *If $L^*v = 0$ or if $L(vu) = 0$, then v is constant.*

The proposition follows immediately from the formula

$$\int_{S^n} vL(uv)\,d\theta = -\int_{S^n} huc_{ij}v_iv_j\,d\theta - \int_\Omega \beta|\nabla\overline{v}|^2$$

where $\overline{v}$ is the harmonic extension of v to Ω and β is the harmonic function in Ω with boundary values uh. Indeed, $L^*v = 0$ and $L(vu) = 0$ both imply that the left-hand side is zero. But the right-hand side only vanishes when v is constant.

In addition to proving that T is open, the proposition contains a uniqueness result: the only infinitesimal changes that preserve R are dilations. This is known as infinitesimal rigidity.

The fact that T is closed depends on *a priori* inequalities. We need to show that given a strictly convex, $C^{k+2,\alpha}$ domain Ω, the $C^{k+2,\alpha}$ norm and modulus of strict continuity are controlled by the $C^{k,\alpha}$ norm and positive lower bound of R. Then a standard limiting argument shows that T is closed.

First of all, we dilate Ω so that B_1 is the smallest ball containing Ω.

PROPOSITION 2. *There exists $r > 0$, depending only on the lower bound for R such that $B_r \subset \Omega \subset B_1$.*

This can be proved by an easy argument involving comparison with a hemisphere.

Consider a function $w: D \to \mathbf{R}$, $D \subset \mathbf{R}^n$, such that $\{(w(x), x_1, \dots, x_n): x \in D\}$ is a portion of $\partial\Omega$. Because $B_r \subset \Omega \subset B_1$, $|\nabla w|$ is bounded above, that is, we have *a priori* bounds on the Lipschitz constant of w. The normal vector is $\theta = (-1, \nabla w)/\sqrt{1+|\nabla w|^2}$, so that

$$d\theta/d\sigma = (1+|\nabla w|^2)^{-((n+2)/2)} \det(w_{ij}) \qquad (w_{ij} = \partial^2 w/\partial x_i \partial x_j)$$

and our equation is

$$(**) \qquad \det(w_{ij}) = (1+|\nabla w|^2)^{(n+2)/2} h/R.$$

Apart from the factor h, this is the equation of the Minkowski problem. R is a smooth function of θ, so the right-hand side involves only first derivatives of w. The function h is the normal derivative of Green's function. As one can see from $\mathbf{R}^2$ and from the linearized operator L, the factor h should be viewed as a nonlinear first order pseudodifferential operator on w. Its regularity properties are only slightly worse that those of R and $1+|\nabla w|^2$. Our strategy is to derive some regularity for h and deduce some further regularity for w from the Monge–Ampère equation $(**)$. Next, based on new estimates for w, we can improve our regularity estimates for h, and so on.

THEOREM 4. *Let $B_r \subset \Omega \subset B_1$, Ω a smooth, strictly convex domain in $\mathbf{R}^{n+1}$. Suppose that, locally, $\partial\Omega$ is given by the graph of functions w satisfying* $\det(w_{ij}) = f$. *Suppose also that*

$$\max_{x\in E} f(x) \le C_1 \int_{\frac{1}{2}E} f(x)\,dx/\operatorname{vol}(E)$$

for all sets $E = \{x \in \mathbf{R}^n : w(x) \le a\cdot x + b\}$. Then the C^1 modulus of continuity and modulus of strict convexity of w depend only on C_1, r, and n.

Note that Theorem 2 implies that $F = (1+|\nabla w|^2)^{(n+2)/2} h/R$ satisfies the hypothesis of Theorem 4 because $|\nabla w|$ is bounded above and R is bounded above and below by positive constants. Theorem 4 follows from the method used by L. Caffarelli **[1]** to derive the same conclusion from the stronger hypothesis $C_1^{-1} \le f \le C_1$.

As a result of Theorem 4 we have control on the modulus of continuity of the Gauss map and hence of $(1+|\nabla w|^2)^{(n+2)/2}/R$. An application of Theorem 3 implies that our function f satisfies, in addition,

$$(\dagger) \qquad \max_{x\in E} f(x) \le (1+\delta) \int_E f(x)\,dx/\operatorname{vol}(E)$$

with $\delta \to 0$ as $\operatorname{vol}(E) \to 0$.

THEOREM 5. *If, in addition to the hypothesis of Theorem 4, we have* $(\dagger)$, *then the L^p norm of w_{ij} is bounded a priori for all $i, j = 1, \dots, n$ and any $p < \infty$. In particular, the $C^{1,\alpha}$ norm of w is bounded a priori for any $\alpha < 1$.*

Theorem 5 follows from the method of another theorem of Caffarelli **[2]** with the same conclusion as Theorem 5, under the stronger hypothesis that f is positive and continuous.

Once we have $C^{1,\alpha}$ control of w it follows that h has an a priori positive lower bound and h belongs to C^α. From this point on the regularity of