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Approximation of continuously differentiable functions, by J. G. Llavona. North-Holland Mathematics Studies, vol. 130, North-Holland, Amsterdam, 1986, xiv + 241 pp., $48.00. ISBN 0-444-70128-1

The year 1885 was an important year for approximation theory, for in that year Weierstrass and Runge announced well-known approximation theorems bearing their names. It is the 1885 theorem of Weierstrass, asserting the density of polynomials in the real variable in the Banach space $C[a, b]$ where $[a, b]$ is a closed interval, that will concern us in this review. Since then several important extensions of the theorem have been obtained by De la Vallé Poussin **[17]**, Bernstein **[5]**, Stone **[15]**, and Whitney **[18]** and others, by stressing one aspect or another of the classical approximation theorem. The results of the first two mathematicians imply Weierstrass theorem on differentiable functions, which asserts that the space of real valued polynomials on R^n is dense in the space $C^m(R^n)$ of m times continuously differentiable real valued functions on R^n, in the topology τ_u^m of uniform convergence of a function and its first m derivatives on compact subsets of R^n. The well-known theorem of Stone referred to in the literature as the Stone-Weierstrass theorem characterizes dense subalgebras, $\mathscr{A}$, of the algebra $C(X)$ of continuous real valued functions on a compact set X in the uniform topology in terms of separation properties of points in X by functions in $\mathscr{A}$. A corollary of this theorem is the density of polynomials in n variables in the Banach space $C(X)$, where X is a compact subset of R^n which is also referred to as Weierstrass approximation theorem in the literature. Whitney's ideal theorem asserts that the τ_u^m-closure of a subspace $\mathscr{A}$ of $C^m(R^n)$ which is also an ideal is the same as the τ_p^m-closure of $\mathscr{A}$, where τ_p^m is the topology of pointwise convergence of a function and its first m derivatives. The results in this monograph are the outcome of the persistent efforts of several mathematicians in the past two decades, mostly under the inspiration of Nachbin, to extend the preceding theorems to the setting of infinite dimensional Banach spaces. It includes among others the results of Abuabara, Aron, Bombal, Ferrera, Gomez, Guerreiro, Lesmes, Llavona, Nachbin, Prolla, Restrepo, Valdivia, Wells, Wulbert, and Zapata related to approximation of continuously differentiable function on Banach spaces.

Before describing the contents and goals of the monograph under review let us recall some terminology. In what follows E, F are real infinite dimensional Banach spaces, and E', F' are the duals of E and F respectively. To formulate the classical approximation theorem in an infinite dimensional setting we need an infinite dimensional notion of a polynomial mapping. A continuous n-homogeneous polynomial, $n \geq 1$, on E to F is a mapping p on E to F which is a composition of the form $A \circ \Delta_n$, where A is a symmetric n-linear transformation on E^n to F, and Δ_n is the diagonal map on $E \to E^n$. The o-homogeneous polynomials are constant functions on E to F. The vector space of continuous n homogeneous polynomials on E to F is denoted by $P(^nE;F)$. An interesting subspace of $P(^nE;F)$ is the space $P_f(^nE;F)$ generated by collections of the form $\varphi^n \otimes y$ where $\varphi \in E'$, $y \in F$, and $(\varphi^n \otimes y)(x) = \varphi^n(x)y$ for $x \in E$. Let $P(E;F) = \sum_{n=0}^{\infty} P^n(E;F)$, and $P_f(E;F) = \sum_{n=0}^{\infty} P_f(^nE;F)$. The vector spaces $P(E;F)$, and $P_f(E;F)$ are respectively known as the space of polynomials on E to F and the space of finite polynomials on E to F. The space $P(^nE;F)$ is equipped with a norm in a natural way, by defining $\|p\| = \sup\{\|p(x)\| \,|x \in E, \|x\| \leq 1\}$. $P(^nE;F)$ equipped with this norm is a Banach space, and the completion of $P_f(^nE;F)$ in $P(^nE;F)$ is denoted by $P_c(^nE;F)$. It is verified that $P_f(^nE;F) = P_c(^nE;F) = P(^nE;F)$ if and only if E is finite dimensional.

If $\mathscr{F}(E;F)$ is a vector space of continuous F-valued functions on E, and τ is any locally convex topology on E, let $\mathscr{F}_\tau(E;F)(\mathscr{F}_{\tau u}(E;F))$ be the subspace of $\mathscr{F}(E,F)$ consisting of functions, f, such that the restrictions of f to bounded subsets B of E are τ-continuous (uniformly

τ-continuous on B). Of special interest here are the spaces $C^m(E;F)$ of m times continuously differentiable F-valued functions on E, $m \in \mathbf{N}$, and the associated spaces $C^m_w(E;F)$ and $C^m_{wu}(E;F)$, where w signifies the weak topology on E. Thus if $f \in C^m(E;F)$, then the nth differential of f, $d^n f$, $0 \leq n \leq m$, is a continuous mapping on E into $P(^nE;F)$. As usual $C^\infty(E;F) = \bigcap_{m\geq 1} C^m(E;F)$. When F is the real line we denote the function spaces $\mathcal{F}(E;F)$, $\mathcal{F}_\tau(E,F)$, etc. simply as $\mathcal{F}(E)$, $\mathcal{F}_\tau(E)$, etc.

When E is infinite dimensional, the spaces $C^m(E;F)$ may be equipped with different locally convex topologies. The topologies τ^m_u, τ^m_p already introduced here when E is finite dimensional, naturally extend to the infinite dimensional setting. The locally convex topology τ^m_c on $C^m(E;F)$ is determined by the seminorms $f \to \sum\{\|d^j f(x)(y)\| \,|x, y \in C, 0 \leq j \leq m\}$, where C is allowed to vary over compact sets in E. Let $C^m_b(E;F)$ be the subspace of $C^m(E;F)$ of functions f such that the function f and its derivatives $d^j f$, $1 \leq j \leq m$, are all bounded on bounded subsets of E. A useful locally convex space of differentiable functions arises by equipping $C^m_b(E;F)$ with the topology τ^m_b of uniform convergence of a function and its first m derivatives on bounded subsets of E.

Unlike the spaces $P_f(E), P(E), C^m(E)$, the spaces $P_f(E;F), P(E;F)$ and $C^m(E;F)$ are not algebras. The situation is somewhat remedied by the concepts of a polynomial algebra and submodules of $C^m(E;F)$ over $C^m(E)$. A subspace $\mathcal{A}$ of $C^m(E;F)$ is called a polynomial algebra if for every $g \in \mathcal{A}$, and $p \in P_f(F;F)$, the composition $p \circ g \in \mathcal{A}$. The spaces $P_f(E;F), P(E;F)$ and $C^m(E;F)$ are polynomial algebras. It is verified that if $F = R$, any polynomial algebra is an algebra in the usual sense.

In his attempts to generalize Whitney's theorem to subalgebras of topological algebras of differentiable functions along the lines of Stone-Weierstrass theorem, Nachbin made a deep contribution by isolating the necessary separation conditions which ensure the density of subalgebras in certain topological algebras of differentiable functions. For instance, a theorem of Nachbin **[10]** asserts that if $\mathcal{A}$ is a subalgebra of $C^m(R^n)$, generated by a subset G of $C^m(R^n)$, then $\mathcal{A}$ is dense in $C^m(R^n)$ in the topology τ^m_u, if and only if G satisfies the following separation conditions: N(1)G separates points in R^n, N(2) for any $x \in E$ there exists a function $f \in G$ such that $f(x) \neq 0$, and N(3) for $x \in E$, and $v \in R^n$, $v \neq 0$, there is a function $f \in G$ such that $df(x)(v) \neq 0$. An immediate corollary of this theorem is the Weierstrass theorem on differentiable functions.

Let us consider the problem of generalizing Nachbin's version of the Weierstrass theorem to the infinite dimensional setting. A special case of the problem is to discuss the density of $P_f(H)$ in $C^2(H)$ in the τ^2_u topology, where H is an infinite dimensional Hilbert space. As observed by Lesmes **[8]**, and independently by Llavona, let us consider the problem of approximating the function $f\colon H \to R$ defined by $f(x) = (x, x)$ in the τ^2_u topology by functions in $P_f(H)$. A simple computation yields that $d^2 f(x) = 2j$, where j is the canonical isomorphism of H onto H'. However if $\varphi \in P_f(H), d^2\varphi(x)$ is a linear transformation of finite rank on H into H' which therefore cannot belong to the set of isomorphisms of H

onto H' (which is an open subset of the Banach space of operators on H into H'). Thus f is not approximable by finite polynomials in the τ_p^2 topology on $C^2(H)$. Thus $P_f(H)$ is not dense in $C^2(H)$ in the τ_u^2 topology. Hence it is a nontrivial problem to extend Nachbin's theorem to the infinite dimensional setting. Note that $P_f(H)$ does indeed satisfy Nachbin's conditions N(1) to N(3). To remedy the situation we may consider a new topology on $C^m(E;F)$. Efforts in this direction by Llavona **[9]**, Bombal and Llavona **[9]**, Prolla **[13]**, Prolla and Guerreiro **[14]** culminated in the approximation theorem that $P_f(E;F)$ is dense in τ_c^m topology if E has approximation property, Day **[7]**. On the other hand Aron and Prolla **[3]** continued their study with the τ_u^m topology by restricting their attention to the important subspace $C_{wu}^m(E;F)$ of $C^m(E;F)$ and succeeded in proving that $P_f(E;F)$ is dense in $C_{wu}^m(E;F)$ in the τ_u^m topology if E' has the bounded approximation property, **[7]**. In fact, their theorem asserts that $P_f(E;F)$ is τ_b^m dense in $C_{wu}^m(E;F)$, if E' has the bounded approximation property. In particular $P_f(H)$ is τ_b^m dense in $C_{wu}^m(H)$, if H is a Hilbert space.

The main goal of the monograph is to discuss the density of various polynomial algebras in diverse spaces of differentiable functions on E to F, which arise by requiring the derivatives to satisfy continuity conditions of one sort or another. These assertions and their proofs are much deeper than the one presented in the preceding paragraph by way of illustration. The author also discusses some important properties of locally convex spaces of differentiable functions on E in terms of properties of E. For instance it is proved that the locally convex space $(C^m(E), \tau_c^m)$ has the approximation property if and only if E has the approximation property. In addition to these theorems, in his attempts to render the book self-contained, the author has elegantly presented recent results on weakly continuous functions and compact holomorphic mappings from the papers of Aron-Herves-Valdivia **[2]** and Aron-Schottenloher **[4]**. Apart from their importance in approximation theory, these theorems are of considerable interest in the larger context of functional analysis in general. An account of the reviewer's theorem characterizing super reflexive spaces, Sundaresan **[16]**, and related theorems on approximation of differentiable functions is also included. The book concludes with a discussion of extending the Paley-Wiener-Schwartz characterization of Fourier transforms of distributions with compact support to the infinite dimensional setting, Abuabara **[1]**, Nachbin and Dineen **[12]**. A proof of Whitney's ideal theorem is presented in the appendix.

This book is mostly self-contained and may be used for a one semester course on applications of functional analysis and Banach space theory for second year graduate students.

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