BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 1, July 1990

SPLITTINGS OF SURFACES

RICHARD K. SKORA

Let F be a compact 2-manifold without boundary and with Euler characteristic $\chi(F) < 0$. Only for convenience endow F with a fixed hyperbolic structure, i.e., a discrete, faithful representation of the fundamental group $\pi_1 F$ into the space of isometries of hyperbolic 2-space. *Teichmüller space*, $\mathscr{T}(F)$, is the space of all hyperbolic structures on F divided out by conjugation. W. P. Thurston [Th1] showed that $\mathscr{T}(F)$ admits a compactification as a ball of dimension $-3\chi(F)$. There is a natural identification of the interior of the ball with $\mathscr{T}(F)$ and the boundary of the ball with the space of projective measured geodesic laminations on F (defined below).

J. W. Morgan and P. B. Shalen [MS1, Mo] considered a more general problem. Let Γ be a finitely generated, nonvirtually Abelian group and let $\mathscr{D}_n = \mathscr{D}(\Gamma, \mathrm{Isom}(H^n))$ be the space of discrete, faithful representations of Γ into the group of isometries of hyperbolic n-space divided out by conjugation. They showed that $\mathscr{D}_n$ admits a compactification $\widehat{\mathscr{D}_n}$ where each point of $\widehat{\mathscr{D}_n} - \mathscr{D}_n$ corresponds to a small action of Γ on an $\mathbf{R}$-tree. When $\Gamma = \pi_1 F$ and $n = 2$, they too show that their boundary $\widehat{\mathscr{D}_n} - \mathscr{D}_n$ is homeomorphic to the space of projective measured geodesic laminations on F.

An $\mathbf{R}$-*tree* is a metric space (T, d), such that any two distinct points are joined by a unique arc and every arc is isometric to an interval in $\mathbf{R}$. It is understood that if a group acts on an $\mathbf{R}$-tree, then it acts by isometries and there is no invariant, proper subtree. An action is *small* if the stabilizer of each arc does not contain a free group of rank two.

The above results motivate studying small actions of Γ on $\mathbf{R}$-

Received by the editors June 28, 1989 and, in revised form, December 20, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30F25; Secondary 32G15, 22E40, 20F32.

Key words and phrases. Surface, hyperbolic structure, lamination, tree.

Supported in part by a SUNY Research Development Grant and by a National Science Foundation Postdoctoral Research Fellowship.

©1990 American Mathematical Society
0273-0979/90 $1.00 + $.25 per page

trees. When $\Gamma = \pi_1 F$, we completely characterize small actions and answer a question from [Sh]. Recall that the only small subgroups of $\pi_1 F$ are cyclic groups. Here is our main theorem (definitions follow immediately). It has a generalization for a compact 2-manifold with boundary.

Theorem. *Let $\pi_1 F \times T \to T$ be an action on an* $\mathbf{R}$*-tree. Then $\pi_1 F \times T \to T$ is dual to a measured geodesic lamination if and only if the stabilizer of each arc is cyclic.*

A *geodesic lamination* $\mathcal{L}$ is a closed subset of F, such that each path component is a simple geodesic. A geodesic lamination is *discrete* if it is a finite union of simple closed geodesics. Say an arc in F is *transverse* to $\mathcal{L}$ if its endpoints lie on the complement of $\mathcal{L}$ and it is transverse locally. A *transverse measure* μ is a function from the set of transverse arcs to the set $[0, +\infty)$, such that (i) $\mu(\gamma + \gamma') = \mu(\gamma) + \mu(\gamma')$; and (ii) $\mu(\gamma) = \mu(\gamma')$, whenever γ, γ' differ by a 1-parameter family of transverse arcs. One may think of a transverse measure on a discrete geodesic lamination as simply an assignment of weights to each geodesic. The set of measured discrete geodesic laminations is dense in the space of measured geodesic laminations [Th1].

Let $\mathbf{H}^2 \to F$ be the universal covering. Given a measured geodesic lamination $(\mathcal{L}, \mu)$ in F its preimage in $\mathbf{H}^2$ is $(\widetilde{\mathcal{L}}, \tilde{\mu})$. Say the action $\pi_1 F \times T \to T$ is *dual* to $(\mathcal{L}, \mu)$ if there is an equivariant, locally constant map $p: \mathbf{H}^2 - \widetilde{\mathcal{L}} \to T$, such that $\tilde{\mu}(\gamma) = d(p(\gamma(0)), p(\gamma(1)))$, for every transverse arc $\gamma: [0, 1] \to \mathbf{H}^2$ meeting each path component of $\mathcal{L}$ at most once.

Morgan and J.-P. Otal [MO] proved the above theorem under an additional geometric hypothesis (cp. [Sk]). And H. Gillet and P. B. Shalen [GS] proved it under the additional hypothesis that the action has rank equal to 1 or 2.

The techniques of J. Stallings [MS1] prove the theorem when the $\mathbf{R}$-tree is a simplicial tree. In this case it has the following interpretation. The Bass–Serre theory [Se] implies that the action on the simplicial tree gives a *splitting* of $\pi_1 F$, e.g., a free product with amalgamation or HNN-extension. And the lamination which will be discrete is called a *splitting* of F.

1. Degenerations of hyperbolic structures on surfaces

Let $\rho \in \mathcal{D}_n = \mathcal{D}(\pi_1 F, \mathrm{Isom}(H^n))$. Define its *length function* $l: \pi_1 F \to \mathbf{R}$ by $l(g) = \inf_{x \in \mathbf{H}^n} d(x, \rho(g)(x))$. Now form the projective space $\mathscr{P} = [0, +\infty)^{\pi_1 F} - 0/\sim$. One gets a map $\Theta : \mathscr{D}_n \to \mathscr{P}$ by sending a representation to its projectivized length function.

Let $\pi_1 F \times T \to T$ be an action on an **R**-tree. Define its *length function* $l: \pi_1 F \to \mathbf{R}$ by $l(g) = \inf_{x \in T} d(x, g(x))$. A small action of $\pi_1 F$ is determined by its length function [CM]. The *space of projective classes of small length functions on trees* $\mathscr{SLF}(\pi_1 F)$ is the image in $\mathscr{P}$ of all small actions on **R**-trees.

Morgan and Shalen showed that the closure $\overline{\Theta(\mathscr{D}_n)}$ is compact and that $\overline{\Theta(\mathscr{D}_n)} - \Theta(\mathscr{D}_n)$ is a subset of $\mathscr{SLF}(\pi_1 F)$. This leads to the compactification $\widehat{\mathscr{D}}_n$. Identify $\mathscr{D}_n$ with its image in $\widehat{\mathscr{D}}_n$ and let $\partial\mathscr{D}_n = \widehat{\mathscr{D}}_n - \mathscr{D}_n$.

Finally given a measured geodesic lamination $(\mathscr{L}, \mu)$ it has a *length function* $l: \pi_1 F \to R$, where $l(g)$ is the transverse measure of the geodesic representative of g. Again the measured geodesic lamination is determined by its length function [Th1, PH]. The *space of projective measured geodesic laminations* $\mathscr{PML}(F)$ is the image in $\mathscr{P}$ of all measured geodesic laminations.

Thurston showed $\mathscr{PML}(F) = \partial\mathscr{D}_2$. By construction $\partial\mathscr{D}_2 \subseteq \partial\mathscr{D}_n \subseteq \mathscr{SLF}(\pi_1 F)$. And the main theorem implies

$$\mathscr{SLF}(\pi_1 F) \subseteq \mathscr{PML}(F).$$

Theorem. *For all* $n \geq 2$, $\partial\mathscr{D}_n = \mathscr{PML}(F)$. ■

The above theorem in the cases $n = 2, 3$ was first proved by Thurston [Th2].

2. Small actions of surface groups on R-trees

The main theorem also has applications to the study of surface group actions on **R**-trees. From [Ha] or [MS2] $\mathscr{PML}(F) \subseteq \mathscr{SLF}(\pi_1 F)$. Again by the main theorem $\mathscr{SLF}(\pi_1 F) \subseteq \mathscr{PML}(F)$. The following answers a question of M. Culler and J. W. Morgan [CM] in the case the group is $\pi_1 F$.

Theorem. $\mathscr{SLF}(\pi_1 F) = \mathscr{PML}(F)$. ■

Since every measured geodesic lamination is approximable by a measured discrete geodesic lamination, the above theorem tells us that every small action of $\pi_1 F$ on an **R**-tree is approximable by a small action on a simplicial tree. It is an open question whether every (small) action of a finitely generated group on an **R**-tree is aproximable by a (small) action on a simplicial tree [Sh].

Finally we may deduce two finiteness results. Fix a small action $\pi_1 F \times T \to T$ and the dual measured geodesic lamination $(\mathscr{L}, \mu)$. A *vertex* of T is a point x, where $T - \{x\}$ has more than two connected components. The vertices of T correspond to the connected components of $\mathbf{H}^2 - \widetilde{\mathscr{L}}$. An area calculation shows the number of orbits of vertices is no greater than $-2\chi(F)$.

The *rank* of the action is the dimension of $G \otimes \mathbf{Q}$ as a vector space over $\mathbf{Q}$, where G is the subgroup of $\mathbf{R}$ equal to $\langle l(g) \rangle_{g \in \pi_1 F}$. Again referring back to the lamination, one sees that the rank is no greater than one plus the dimension of $\mathscr{PML}(F)$ which is $-3\chi(F)$.

3. Sketch of the proof of the main theorem

Suppose that $\pi_1 F \times T \to T$ is dual to a measured geodesic lamination. Then the stabilizer of each arc in T is contained in the fundamental group of a path component of the lamination which is cyclic.

Now conversely suppose the action has cyclic arc stabilizers. The starting point is a theorem of A. Hatcher [Ha] or Morgan and Otal [MO]. They prove that there is an action on an **R**-tree $\pi_1 F \times R \to R$ and an equivariant morphism $\phi: R \to T$, such that $\pi_1 F \times R \to R$ is dual to a measured geodesic lamination $(\mathscr{L}, \mu)$ on F.

A *morphism* from R to T is a map $\phi: R \to T$, such that for each arc $[x, y]$ in R there is an arc $[x, z] \subseteq [x, y]$, such that $\phi|[x, z]$ is an isometry. The morphism ϕ *folds* at a point $x \in R$ if there are arcs $[x, y]$ and $[x, y']$ such that $[x, y] \cap [x, y'] = \{x\}$; $\phi|[x, y]$ and $\phi|[x, y']$ are embeddings; and $\phi([x, y]) = \phi([x, y'])$. A morphism either is a monomorphism or folds at some point. Thus it suffices to show that ϕ does not fold.

We will prove the theorem by contradiction. Suppose ϕ folds at x. Let $[x, y]$ and $[x, y']$ be as in the definition of fold. We may suppose x is a vertex.

Suppose R is a simplicial tree. Then $\mathscr{L}$ is a discrete geodesic lamination and up to rechoosing we may suppose $[x, y]$, $[x, y']$ have infinite cyclic stabilizers $\langle g \rangle$, $\langle g' \rangle$, respectively. It is easy to see from the geometry of $\mathscr{L}$ that $\langle g \rangle$, $\langle g' \rangle$ are conjugate, but $\langle g, g' \rangle$ is free of rank two. Therefore the stabilizer of $\phi([x, y]) = \phi([x, y'])$ contains this free group of rank two which is a contradiction.

Now for R a general $\mathbf{R}$-tree the proof proceeds by studying $(\mathscr{L}, \mu)$ more carefully. An important tool is a train track. A *train track* is a smooth subgraph τ of F, such that the double of each component of $F - \tau$ along its smooth frontier has negative Euler characteristic. An important combinatorial property of τ is that every smoothly immersed curve γ in τ is determined by the lift to $\mathbf{H}^2$ of its initial and terminal points [Th1, PH]. In particular, every smoothly immersed loop is nontrivial. Say that a lamination is *carried* by a train track τ if there is a map $f\colon F \to F$ fixed on τ and homotopic to the identity, such that f composed with each smooth curve in $\mathscr{L}$ is a smooth immersion. Every geodesic lamination is carried by a train track [Th1, PH].

The second tool is an interval exchange map. Let α be transverse to $\mathscr{L}$ with lift $\tilde{\alpha}$ in $\widetilde{\mathscr{L}}$, such that the image of $\tilde{\alpha}$ in R is $[x, y]$ and $\mu(\alpha) = d(x, y)$. Fix an orientation and transverse orientation on α and let I be an interval of length $\mu(\alpha)$. Then parallel translation of α along $\mathscr{L}$ determines an interval exchange map $A\colon I \to I$. If we identify $[x, y]$ with I, then there are a finite number of elements $g_1, \ldots \in \pi_1 F$ which permute the subarcs of $[x, y]$ exactly the same way A permutes the subintervals of I. It follows that words of length n in $g_1, \ldots$ permute the subarcs of $[x, y]$ exactly the same way A^n permutes the subintervals of I.

So corresponding to both $[x, y]$, $[x, y']$ are interval exchange maps A, A' respectively. Or equivalently, there are group elements $g_1, \ldots$ and $g_1', \ldots$ which permute subarcs of $[x, y]$ and $[x, y']$, respectively. Let τ carry $\mathscr{L}$. Up to passing to a finite fold covering and rechoosing $[x, y]$ and $[x, y']$ we may suppose that distinct positive words in $g_1, \ldots, g_1', \ldots$ are represented by distinct smoothly immersed loops in τ.

Since $[x, y]$ and $[x, y']$ have identical images in T, we should consider the way $g_1, \ldots, g_1', \ldots$ permute the subarcs of $\phi([x, y])$. For any $z \in \phi([x, y])$, let $\mathscr{C}_n$ be the set of sequences $\langle h_1, \ldots, h_n \rangle$, such that $h_i \in \{g_1, \ldots, g_1', \ldots\}$ and $h_i \circ \cdots \circ h_1(z) \in \phi([x, y])$, for all i. The set $\mathscr{C}_n$ grows exponentially with n.

However, A, A' are defined by finitely many translations. So their n-fold compositions are defined by a certain number of translations which grows polynomially with n. Therefore $\mathscr{C}_n(z)$ grows polynomially with n. Now one may argue for all but at most