a useful way to get students into research. And, these nonexperts are my greatest concern. The intricacies of nuclear spaces or, more generally Kothe sequence spaces, are beyond the grasp of beginners, at least any I have encountered.

Thus, to reiterate, I see the book under review as a general reference book for experts and advanced students and like any general reference, the book has some value. The book contains a lot of material. Unfortunately, I felt that I had read most of it years before. In the words of the great philosopher Yogi Berra, reading this was déjà vu all over again.

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J. R. RETHERFORD
LOUISIANA STATE UNIVERSITY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 1, July 1990

0273-0979/90 $1.00 + $.25 per page

Continuous decoupling tranformations for linear boundary value problems, by P. M. van Loon. Centrum voor Wiskunde en Informatica, CWI Tract #52, Stichting Mathematisch Centrum, Amsterdam, 1988, vi + 198 pp. ISBN 90-6196-353-2

This book deserves more readers than its title is likely to attract. Only specialists who have already some familiarity with the subject will realize immediately that these boundary value problem concern *ordinary* differential equations of any order n, that "decoupling" means that the differential equation is transformed into a pair of differential equations, each of lower order than n and that the transformations are called continuous, because they deal with the differential equations themselves, not with discrete approximations.

I have been trying to think of a better short descriptive title, but without success. The computational and theoretical difficulties this book comes to grips with are present in a majority of boundary value problems of interest in the applications. Nevertheless, an inviting sounding title such as "Numerical solution of boundary value problems for linear ordinary differential equations" would be too comprehensive and therefore misleading.

To describe here these difficulties in simple, though admittedly very vague, terms it suffices to concentrate on *homogeneous* differential equations written in the standard form

$$\frac{dx}{dt} = A(t)x,$$

where $A(t)$ is an $n \times n$ real matrix and x an n-dimensional vector depending on t. The solutions are n-dimensional vector functions of t. If one knows a fundamental system of solution vectors, i.e., n solutions that are linearly independent at one point—and, hence, everywhere—then all solutions are linear combinations of those n vectors. Now, it turns out that the vectors of a fundamental system differ very much in size. Even if they are normalized so as to be, say, all of length one at one particular point some will change with t much faster—in length as well as in direction—than others. Therefore, which of those vectors are numerically important in a linear combination may differ radically from point to point. This phenomenon is likely to ruin the accuracy of computational schemes for any but very simple boundary value problems, unless the procedure is sophisticated enough to cope with this problem.

Parallel with the rapid growth of power of the available computing equipment in the past thirty years, there has been a great deepening and broadening of the mathematical analyses pertinent to the question just described. In 1958, J. L. Massera and J. J. Schäffer introduced the useful concept of "dichotomy" of a fundamental system of solutions, and a few years later W. A. Coppel made a thorough study of its implications. Roughly speaking, a dichotomy is present if the set of all solutions can be split into two subsets, one of which contains only solutions that grow with t in the interval considered, while those in the other subset shrink with growing t. There are two types of problems in which this dichotomy is most pronounced. One occurs when t is very large (or close to a singularity). Then, under reasonable smoothness assumptions, the classical asymptotic theory implies that the dichotomy is "exponential", i.e., that there is a fundamental system of n vectors some of which grow as fast as exponential functions, as $t \to \infty$; the "dominant" solutions, while the other, the "dominated" solutions shrink or, at worst, grow not faster than some power of t.

The second type of problems in which dichotomy is a useful concept consists of differential equations in which some coefficients are "very large." Nowadays, one often meets the term "stiff" differential equations in the study of such problems. The origin of this terminology is, presumably, the simple differential equation that describes movements caused by a stiff spring. Often the imprecise words "very large" can be replaced with a mathematically more satisfactory description by introducing a large parameter into the differential equation. The task is then to study the solutions as that parameter grows to infinity. In this way one enters the well-developed subject of the asymptotic theory of differential equations as a parameter tends to infinity. It contains, in particular, the topic called "singular perturbations."

The "decoupling" referred to in the title of the book consists of a linear transformation $x = T(t)y$ of the unknown vector function x in the given differential equation

$$\frac{dx}{dt} = A(t)x + f(t),$$

for which the coefficient matrix $\widetilde{A}$ of the resulting new differential equation

$$\frac{dy}{dt} = \widetilde{A}(t)y + \widetilde{f}(t)$$

has the special, block-diagonal, form

$$\widetilde{A}(t) = \begin{bmatrix} \widetilde{A}_{11}(t) & \widetilde{A}_{12}(t) \\ 0 & \widetilde{A}_{22}(t) \end{bmatrix}.$$

If $\widetilde{A}_{11}$ has $k < n$ rows and columns, the task of solving the differential equation has, thus, been decomposed into solving first a system of $n-k$ equations and then one of order k. The boundary conditions mentioned in the title of the book are of the form

$$B^0x(0) + B^1x(1) = b$$

with constant $n\times n$ matrices B^0, B^1, and a vector b. (The choice of the interval as $[0, 1]$ is for convenience only.)

If the solution space of the corresponding homogeneous system of differential equations has a dichotomy, it would be very convenient if the transformation matrix T decoupled the system into one for the dominated solutions and one for the dominating ones. However, k is not known in advance, nor does one know which of the infinitely many decoupling transformations T achieve such a separation. Thus, it seems that the search for a "good" decoupling is a case of begging the question.

On the other hand, even the incomplete insight obtainable by a theoretical analysis is a great help in the setting up of an effective computational scheme. One aim of the book is to give a connected account of this material, including careful proofs wherever the properties to be used are mathematical theorems and supplying heuristic or numerical justifications wherever no complete error analysis is available.

The reader is expected to be familiar with the elementary theory of matrices. Many interesting properties beyond that material are derived in the first two chapters of the book. It is also assumed that the reader knows already some of the standard computational techniques such as the Runge-Kutta method. The presentation is clear but so concise that most prospective readers will have to put in some effort to fill in all the gaps.

The great power of modern computing equipment is taken full advantage of. For instance, the requirement that the transformation with matrix $T(t)$ produce a block-diagonal new coefficient matrix represents a system of quadratic differential equations. That system possesses a large family of solutions. The size of the system can be somewhat reduced by additional restrictions, but even that system is often rather large. It goes by the name of Riccati equation in gneralization of the well-known simple scalar case.

After the original system has thus been block-diagonalized, the numerical solution of the boundary value problem usually requires a "multiple shooting" procedure to avoid accumulation of errors. This term describes a subdivision of the interval at whose end-points the boundary conditions are prescribed into a set of subintervals. The intermediary boundary values are connected by linear conditions. The total number of parameters that have to be simultaneously determined in this way may be quite large, even if strategies such as the not too clearly defined "invariant imbedding" are applied.

The author describes carefully the detailed steps of the computations that represent the "bottom line" of his investigations. Their complexity exceeds what existing computing equipment would have been capable of coping with, only a few decades ago. A number of illustrative examples is included to verify the feasibility of the author's methods.

An extensive literature on the subject of this book is listed and quoted. The bibliography has sixty-three entries, most of them less than fifteen years old. In addition, a number of the results appears to be due to the author himself. The strongest influence is the work of R. M. M. Mattheij.

The longest chapter discusses stiff problems, particularly those of singular perturbation type. The author shows that his techniques can produce numerically satisfactory answers for boundary value problems whose solutions are nearly discontinuous in boundary layers or at certain interior points. He hints that even well-studied difficult problems of fluid dynamics, such as the boundary value problems for the Orr–Sommerfeld equations that govern the onset of turbulence in flows with large Reynolds number, can be handled in his way, but no applications to such important practical problems are included in the book.

Singular perturbation problems have long been explored by analytical methods based on asymptotic expansions or else, on the theory of differential inequalities. I am wondering if those techniques could not be combined to advantage with the approach of this book.

For an analogous situation such a combination is, indeed, presented in the last chapter. There, the author develops a method for the numerical solution of boundary value problems with analytic coefficients, when one endpoint of the interval considered is a singularity of the first kind. The well-known analytic theory for such singular points is a useful ingredient in the proposed procedure. The dichotomy of the solution space is here apparent from the outset. Most of the content of this chapter is a condensed version of an earlier report by the author. ("Reducing a singular linear two-