

REFERENCES

1. M. Artin, A. Grothendieck and J. L. Verdier, *Théorie des topos et cohomologie étalée des schémas*, Lecture Notes in Math., vol 269, Springer-Verlag, Berlin and New York, 1972.
2. J. Benabou, *Structures algébrique dans les catégories*, Cahiers Topologie Géom. Différentielle Catégoriques **10** (1968), 1–126.
3. R. Brown and T. Porter, *Mathematics in context, a new course*, U. C. N. W. Maths Preprint 90.09, School of Mathematics, The University of Wales, Bangor, Wales.
4. Y. Diers, *Catégories localement multiprésentables*, Arch. Math. **34** (1980), 344–356.
5. C. Ehresmann, *Esquisses et types de structures algébriques*, Bull. Instit. Polit., Iasi **14** (1968), 1–14.
6. P. Gabriel and F. Ulmer, *Lokal präsentierbare Kategorien*, Lecture Notes in Math., vol. 221, Springer-Verlag, Berlin and New York, 1971.
7. R. Guitart et C. Lair, *Calcul syntaxique des modeles et calcul des formules internes*, Diagrammes **4** (1980), 1–106.
8. —, *Limites et co-limites pour représenter les formules*, Diagrammes **7** (1982), GL1–GL24.
9. J. Isbell, *Small adequate subcategories*, J. London Math. Soc. **43** (1968), 242–246.
10. —, *General functorial semantics, I*, Amer. J. Math. **94** (1972), 535–596.
11. C. Lair, *Catégories modelables et catégories esquissables*, Diagrammes **6** (1981), L1–L20.
12. F. W. Lawvere, *Functorial semantics of algebraic theories*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 869–872.
13. D. Lazard, *Sur les modules plats*, Comp. Rend. Acad. Sci. Paris **258** (1964), 6313–6316.
14. M. Makkai and G. Reyes, *First order categorical logic*, Lecture Notes in Math., vol. 611, Springer-Verlag, Berlin and New York, 1977.
15. F. Ulmer, *Bialgebras in locally presentable categories*, University of Wuppertal, 1977, preprint.

JOHN W. GRAY

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

BULLETIN (New Series) OF THE
 AMERICAN MATHEMATICAL SOCIETY
 Volume 25, Number 1, July 1991
 ©1991 American Mathematical Society
 0273-0979/91 \$1.00 + \$.25 per page

The volume of convex bodies and Banach space geometry, by Gilles Pisier. Cambridge Tracts in Mathematics, vol. 94, Cambridge University Press, Cambridge, 1989, xv + 250 pp., \$49.50. ISBN 0-521-364655

A convex body is a compact convex subset of \mathbf{R}^n with a non-empty interior. The study of convex bodies has deep roots in the

ancient world, especially in the mathematical work of Archimedes and his predecessors. It now has close ties to number theory, probability theory, harmonic analysis, approximation theory, and a host of other subjects.

Consider a symmetric convex body $B \subset \mathbf{R}^n$, the symmetry being with respect to the origin. In most of the classical work, B plays the leading role, but in more recent work, it is often the dimension n . For example, Fritz John (1948) proved that for every such B there is an ellipsoid D such that

$$D \subset B \subset \sqrt{n}D.$$

In general, the number \sqrt{n} cannot be replaced by a smaller number: consider the n -dimensional cube $[-1, 1]^n$.

Dvoretzky (1961) discovered that much sharper ellipsoidal bounds exist for some k -dimensional sections of B provided that k is small relative to n . In fact, to each positive number ε there corresponds a positive number δ such that if

$$(1) \quad 1 \leq k \leq \delta \log n,$$

then there is a k -dimensional subspace F of \mathbf{R}^n and a k -dimensional ellipsoid $D \subset F$ such that

$$D \subset B \cap F \subset (1 + \varepsilon)D.$$

Thus, for small ε , the section $B \cap F$ is almost ellipsoidal. The number $\delta = \delta(\varepsilon)$ can be chosen independently of n and B . This beautiful discovery has led to many others.

Dvoretzky's proof yields a factor with a smaller order of magnitude than $\log n$ in (1). Milman (1971) gave a different proof using an isoperimetric inequality on the sphere and related work of Paul Lévy to obtain $\log n$, which has the best possible order of magnitude as $n \rightarrow \infty$. See also the fundamental paper of Figiel, Lindenstrauss, and Milman (1977) and the references given there.

Gordon (1985, 1988) used his extension of Slepian's inequality for Gaussian processes, which now has a short proof due to Kahane (1986), to give a proof of Dvoretzky's theorem that yields a positive number β such that the optimal choice of $\delta(\varepsilon)$ satisfies $\delta(\varepsilon) \geq \beta \varepsilon^2$ for all $\varepsilon \in (0, 1]$. If a question of Knaster (1947) has an affirmative answer, then, as Milman (1988) has shown, (1) can be replaced by

$$1 \leq k \leq \alpha(n, \varepsilon) \log n / \log 1/\varepsilon$$

where $\alpha(n, \varepsilon) \rightarrow 1$ as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$.

Earlier, Mahler (1939) had considered the product $|B| \cdot |B^\circ|$ of the volume of a symmetric convex body B and the volume of its polar

$$B^\circ = \{x \in \mathbf{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in B\}$$

where $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$. The volume product $|B| \cdot |B^\circ|$ is a linear invariant. If $u : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is linear and invertible, then $|u(B)| \cdot |u(B)^\circ| = |B| \cdot |B^\circ|$. In particular, if

$$D_n = \{x \in \mathbf{R}^n : \langle x, x \rangle \leq 1\}$$

and D is any symmetric ellipsoid, then $D = u(D_n)$ for such a map u so

$$|D| \cdot |D^\circ| = |D_n| \cdot |D_n^\circ| = |D_n|^2.$$

Mahler proved that the volume product satisfies

$$(2) \quad 4^n (n!)^{-2} \leq |B| \cdot |B^\circ| \leq 4^n$$

and gave applications of this inequality to number theory. Except for $n = 1$, neither side of (2) is sharp. Santaló (1949) found the sharp bound for the right-hand side:

$$(3) \quad |B| \cdot |B^\circ| \leq |D_n|^2.$$

Recent proofs of (3) are contained in the papers of Saint-Raymond (1981) and of Meyer and Pajor (1989).

In his 1939 paper, Mahler had conjectured that the sharp lower bound of $|B| \cdot |B^\circ|$ is $4^n/n!$, which is the value of the volume product for $B = [-1, 1]^n$. If this conjecture is correct, then there is a positive number α such that

$$(4) \quad \alpha \leq \left(\frac{|B| \cdot |B^\circ|}{|D_n|^2} \right)^{1/n} \leq 1$$

for all symmetric convex bodies $B \subset \mathbf{R}^n$ and all $n \geq 1$. Although the correctness of Mahler's conjecture has not yet been decided, Bourgain and Milman (1987) have recently established the existence of such a positive number α . Their methods differ greatly from those used by Mahler and Santaló and rest partly on some of the ideas developed in the extensive exploration of Dvoretzky's theorem and other parts of the local theory of Banach spaces.

The local theory is the study of Banach spaces through their subspaces of finite dimension. Symmetric convex bodies play a key

role in this study: if B_X is the closed unit ball of a Banach space X and E is a finite-dimensional subspace of X , then $B_E = B_X \cap E$ can be identified with a symmetric convex body, and of course every symmetric convex body is the closed unit ball of a finite-dimensional Banach space.

If an infinite-dimensional Banach space X has the property that, for some integer $k \geq 2$, all k -dimensional subspaces of X are isometric, is the space X isometric to a Hilbert space? This is one of the many questions posed by Banach in his book (the 1987 edition contains a discussion by Pełczyński and Bessaga of the current status of these questions as well as a survey of some related work). Dvoretzky's theorem yields an affirmative answer. It also confirms the conjecture of Grothendieck that any separable Hilbert space can be finitely represented in every infinite-dimensional Banach space with the same field of scalars. For the real field, see Dvoretzky (1961).

Pisier's book is a valuable guide to some of the recent progress in the local theory of Banach spaces. The presentation is largely self-contained. Most of the material after the first hundred pages of background appears here in a book for the first time. One of the key theorems proved is the Bourgain-Milman inequality (the left-hand side of (4) above). Another is the earlier quotient-of-a-subspace theorem of Milman (1985). Still another is Milman's inverse of the Brunn-Minkowski inequality (1986). Clearly, a complete inverse to the Brunn-Minkowski inequality does not exist but Milman has discovered a partial inverse that has surprising consequences. In fact, it can be used to prove the other two theorems.

In this book, Pisier uses such tools as type, cotype, and K -convexity. Their main source is the paper of Maurey and Pisier (1976) and later papers such as Pisier (1982). He also uses entropy numbers, approximation numbers, Gelfand numbers, Kolmogorov numbers, volume numbers and other analytic, probabilistic, and geometric tools from a variety of sources.

Approximation numbers are used to define the concept of *weak Hilbert space* (cf. Pisier (1988) and the references given there). The Lorentz space $L^{2\infty}(0, 1)$, sometimes denoted by $\text{weak-}L^2(0, 1)$, is *not* a weak Hilbert space. The last third of the book is devoted to the exploration of this concept and the related concepts of weak type and weak cotype. Each has a simple geometric characterization in terms of volume ratios. For example, a Banach space X is a weak Hilbert space if and only if there is a real

number γ such that if F is an n -dimensional subspace of X , then there are ellipsoids D , $G \subset F$ satisfying

$$D \subset B_X \cap F \subset G \quad \text{and} \quad (|G|/|D|)^{1/n} \leq \gamma.$$

Weak Hilbert spaces have a number of other interesting properties.

For example, consider a complex Banach space X and a nuclear operator $T : X \rightarrow X$. If X is a Hilbert space, or is isomorphic to a Hilbert space, then the eigenvalues of T are absolutely summable. See Johnson, König, Maurey, and Retherford (1979). If X is a weak Hilbert space, the sequence (λ_k) of eigenvalues of T , repeated according to their multiplicity and arranged so that $|\lambda_1| \geq |\lambda_2| \geq \dots$, need not be absolutely summable but does satisfy the weaker condition

$$(5) \quad \sup_k k |\lambda_k| < \infty.$$

More precisely, the left-hand side is majorized by a constant multiple of the nuclear norm of T . In fact, (5) characterizes the weak Hilbert spaces among the complex Banach spaces.

This overview gives only a glimpse of the contents of this substantial and important book.

REFERENCES

- S. Banach, *Theory of linear operations*, with a supplement by A. Pelczyński and C. Bessaga on *Some aspects of the present theory of Banach spaces*, North-Holland, Amsterdam, 1987.
- J. Bourgain and V. D. Milman, *New volume ratio properties for convex symmetric bodies in \mathbf{R}^n* , *Invent. Math.* **88** (1987), 319–340.
- A. Dvoretzky, *Some results on convex bodies and Banach spaces*, Proceedings of the International Symposium on Linear Spaces held at the Hebrew University of Jerusalem, July 5–12, 1960, The Israel Academy of Sciences and Humanities, Jerusalem, 1961, pp. 123–160.
- T. Figiel, J. Lindenstrauss, and V. D. Milman, *The dimension of almost spherical sections of convex bodies*, *Acta Math.* **139** (1977), 53–94.
- Y. Gordon, *Some inequalities for Gaussian processes and applications*, *Israel J. Math.* **50** (1985), 265–289.
- , *Gaussian processes and almost spherical sections of convex bodies*, *Ann. Probab.* **16** (1988), 180–188.
- F. John, *Extremum problems with inequalities as subsidiary conditions*, Studies and Essays presented to R. Courant on his 60th birthday, January 8, 1948, Interscience, New York, 1948, pp. 187–204.
- W. B. Johnson, H. König, B. Maurey, and J. R. Retherford, *Eigenvalues of p -summing and l_p -type operators in Banach spaces*, *J. Funct. Anal.* **32** (1979), 353–380.

