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Subharmonic functions, vol. 2, by W. K. Hayman. Academic Press, London, 1990, 590 pp., $53.50. ISBN 0-12-334802-1

A function $u : \Omega \to [-\infty, \infty)$, where Ω is a domain in $\mathbb{R}^m$, is said to be subharmonic (s.h.) if it is upper semicontinuous, not identically $-\infty$, and satisfies the sub mean value inequality: its average over the boundary of each ball contained in Ω is greater than or equal to its value at the center. For $m = 1$ the s.h. functions are the convex ones.

S.h. functions were introduced by F. Riesz in the 1920s. They have come to play a central role in several branches of analysis, notably potential and complex function theories. The book under review is a sequel to *Subharmonic functions*, vol. 1, which Hayman co-authored with P. B. Kennedy [HK]. Volume 1 was devoted to development of the rudiments of potential theory, such as solution of the Dirichlet problem for $\Delta u = 0$, and to the basic properties of subharmonic functions, such as the Riesz decomposition theorem. (If u is s.h. in Ω then its distributional Laplacian Δu, known as the Riesz mass, is a locally finite positive measure on Ω, and, loosely speaking, u equals a potential of Δu plus a harmonic function.) The theory expounded there works pretty much the same in $\mathbb{R}^m$ for every $m \geq 2$.

Volume 2, at 590 pages, is twice as long as Volume 1. Its principal aim is to study, in depth, certain families of extremal problems about entire and meromorphic functions of one complex variable. Most of these questions first arose in the early part of the twentieth century. If f is analytic in $\Omega \subset \mathbb{C} = \mathbb{R}^2$ then $\log|f|$ is s.h., while if f is meromorphic then $\log|f|$ is "δ-subharmonic," that is, the difference of two s.h. functions. The problems treated here turn out often to be most naturally posed in the more general s.h. or δ-s.h. context. Thus, the emphasis in Volume 2 is on functions s.h. in all of $\mathbb{C}$, although there are also numerous results for functions in the unit disk of $\mathbb{C}$, as well as some that still hold in $\mathbb{R}^m$ for $m \geq 3$.

One of the book's main themes is the relation between the maximum and minimum values of s.h. functions on circles. Let u be s.h. in $\mathbb{C}$. Define functions $A, B : [0, \infty) \to [-\infty, \infty)$ by

$$A(r) = A(r, u) = \inf_\theta u(re^{i\theta}), \qquad B(r) = B(r, u) = \sup_\theta u(re^{i\theta}).$$

Then, by the maximum principle, $B(r)$ increases as r increases, but the behavior of $A(r)$ is often erratic. For instance, it can be $-\infty$ for some values of r. Nevertheless, if $B(r)$ increases not too rapidly, then there are senses in which the growth of B is controlled by that of A. The most primitive manifestation is when $\lim_{r\to\infty} B(r) < \infty$, so that u is bounded above in $\mathbb{C}$. Then u must be constant and hence $A(r) \equiv B(r)$. This phenomenon is peculiar to two dimensions. For $m \geq 3$ there exist nonconstant s.h. functions u on $\mathbb{R}^m$ which are bounded above, and there exist functions for which $A(r) \equiv -\infty$. Thus, the "A controls B" results discussed below have no immediate analogues in $\mathbb{R}^m$ for $m \geq 3$, although there are kindred results, such as the analogue of "Paley's conjecture," in which $B(r)$ is controlled by a mean value of u on spheres $|x| = r$.

The order λ of a s.h. function u on $\mathbb{C}$ is defined by

$$\lambda = \overline{\lim_{r\to\infty}} \frac{\log B(r)}{\log r}.$$

Following up earlier work by A. Wiman, Littlewood (1908) proved the existence of constants $C(\lambda) > -\infty$ such that if u is s.h. in $\mathbb{C}$ with finite order λ then

$$\overline{\lim_{r\to\infty}} \frac{A(r)}{B(r)} \geq C(\lambda). \tag{1}$$

Littlewood stated his result for functions of the form $u = \log|f|$, with f entire. (Remember, Riesz's introduction of s.h. functions was eighteen years in the future.) His technique still works, though, for general s.h. u. Similar considerations exist for some of the other events appearing in the history below. Henceforth, we shall attribute to various authors s.h. or δ-s.h. statements which they actually made in terms of entire or meromorphic functions. A method developed by B. Kjellberg, P. B. Kennedy, and W. al Katifi for approximating s.h. u by functions $\log|f|$ often enables one to find extremal functions of the latter form for the problems considered here, and thereby shows that the inequalities proved are still sharp within the originally stated context.

Let us return now to Littlewood's inequality (1), and let $C(\lambda)$ denote also the largest possible such constant. Littlewood showed that for $0 \leq \lambda \leq \frac{1}{2}$ we have $C(\lambda) \geq \cos 2\pi\lambda$. He conjectured that for $0 \leq \lambda < 1$ the correct value should be $C(\lambda) = \cos \pi\lambda$. An extremal function would be u_λ, defined for $\theta \in [-\pi, \pi]$ by

$$u_\lambda(re^{i\theta}) = r^\lambda \cos \lambda\theta.$$

Note that for $z \in \mathbb{C}\backslash(-\infty, 0)]$ $u_\lambda(z) = \mathrm{Re}\, z^\lambda$ is harmonic, so that its Riesz mass is supported on the negative real axis. Also, for fixed r, $u_\lambda(re^{i\theta})$ is a symmetric decreasing function of θ. Hence

$$A(r, u_\lambda) = u_\lambda(-r) = (\cos \pi\lambda)u_\lambda(r) = (\cos \pi\lambda)B(r, u_\lambda),$$

so that the ratio $A(r)/B(r)$ is constant when $u = u_\lambda$.

Littlewood's conjecture for $0 \leq \lambda < 1$ was confirmed, independently, by G. Valiron (1914) and by Wiman (1915). When $\lambda = 1$ it is still true that $C(1) = -1$, but the value of $C(\lambda)$ for $\lambda > 1$ remains unknown. Wiman conjectured in 1918 that $C(\lambda) = -1$ for $1 \leq \lambda \leq \infty$. A. Beurling, who had been a student of Wiman, provided positive evidence in 1949 by showing that the lim sup in (1) is indeed ≥ -1 for functions which assume their minima along a ray. Hayman and Kjellberg (1978) extended this to the case of minima along some curve. But Wiman's conjecture had been disproved for large λ by Hayman in 1952. He constructed examples of infinite order for which $A(r)/B(r) \to -\infty$, and others of finite order which show that $\overline{\lim}_{\lambda\to\infty} C(\lambda) = -\infty$. It is possible, but seems unlikely to the reviewer, that $C(\lambda) = -1$ might hold for λ slightly larger than one.

On the subject of genealogy, we note that Kjellberg was a doctoral student of Beurling. Hayman, whose graduate studies were at Cambridge in the late 1940s, was assigned by Littlewood for his thesis the task of constructing a counterexample to the Bieberbach conjecture. As Hayman related in his lecture at a conference in 1985 commemorating de Branges's proof of that conjecture, he did not succeed and never did obtain a Ph.D. Be that as it may, I think it fair to say that Hayman's subsequent body of work, part of which is presented in this book, represents the full flowering of one branch of the British tradition of hard analysis planted by Hardy and Littlewood.

The Wiman-Valiron "$\cos \pi\lambda$ theorem" inspired efforts by Pólya, Denjoy, and others. An account of work up to about 1953 is given in the book *Entire functions*, by R. P. Boas [Bo]. Kjellberg obtained a substantial improvement in 1963. Let u be s.h. in $\mathbb{C}$. We make no a priori assumption about its order. Let λ be a given number in $(0, 1)$.

Kjellberg's Theorem. *Either*

$$\varlimsup_{r\to\infty} [A(r, u) - (cos\ \pi\lambda)B(r, u)] = \infty \tag{2}$$

or else

$$\lim_{r\to\infty} \frac{B(r, u)}{r^{\lambda}} = \alpha$$

exists, and is positive.

The case $\lambda = \frac{1}{2}$ had been proved earlier by Heins. If u in fact does have order $\lambda \in (0, 1)$, we recover the original $\cos\pi\lambda$ theorem by applying Kjellberg's theorem with λ replaced by $\lambda + \varepsilon$ for small ε. Concerning the extreme cases, work by J. M. Anderson (1966), P. Fenton (1976), and others shows that if the $\varlimsup$ in (2) is finite and if $\alpha < \infty$, then u must asymptotically strongly resemble αu_{λ}. For example, the amount of Riesz mass $n(r, u)$ in the disk $|z| < r$ must satisfy $n(r, u) \sim \alpha\frac{\sin\pi\lambda}{\pi}r^{\lambda}$, and most of the mass is supported near a "piecewise radial" set $\{re^{i\theta(r)} : 0 < r < \infty\}$ where $\theta(r) = \theta_n$ for $\theta_n \le r < \theta_{n+1}$, with $\sum_{n=1}^{\infty}(\theta_{n+1} - \theta_n)^2 < \infty$. Such results are called regularity theorems.

There are many other variations on the $\cos\pi\lambda$ theme. For example, P. D. Barry, also in 1963, gave lower bounds for the size of the set of r on which a s.h. function in $\mathbb{C}$ of order $\lambda \in (0, 1)$ must satisfy $A(r, u) \ge (\cos\pi\kappa)B(r, u)$. Here κ is preassigned, with $\lambda < \kappa < 1$. Barry's theorem, which improved one of Besicovitch (1927), was shown via examples by Hayman to be sharp for every allowable λ and κ.

Let us return now to Kjellberg's theorem. We shall outline a proof constructed by M. Essén [E], which uses results of Hellsten-Kjellberg-Norstad [1970]. The general ideas, which go back to the early days of the theory and are illustrative of its typical modus operandi, are three: 1. artistry with integrals, 2. harmonic majorization, and 3. especially, a symmetrization step. In this instance one replaces u by a function v obtained by swinging the Riesz mass of u to the negative real axis.

Suppose first that u is s.h. in $\Delta\backslash(-1, 0]$, where Δ denotes the unit disk, and is continuous on $\overline{\Delta}\backslash\{-1\}$. Then for $r \in (0, 1)$

there is an integral inequality

$$(3)\qquad \begin{aligned} u(r) \le & \int_0^1 [u(t) + u(-t)]L(r, t)dt \\ & + \int_0^{\pi} [u(e^{i\phi}) + u(e^{-i\phi})]H(r, \phi)d\phi, \end{aligned}$$

where L and H are positive kernels.

To prove this, note that a function subharmonic in a half disk is majorized by the harmonic function with the same boundary values. Apply this to u in the upper, lower, and right halves of Δ. Calculation of the half-disk Poisson kernel and iteration lead to (3). If u is harmonic in $\Delta\backslash(-1, 0)]$ then equality holds in (3).

Next, suppose that u is s.h. in Δ, and that $\sup_\Delta u \equiv B(1) < \infty$. Then

$$u(z) = -\int_\Delta g(z, \zeta)d\mu(\zeta) - \int_0^{2\pi} P(z, \phi)d\nu(\phi) + B(1)$$

where g and P are the (positive) Green function and Poisson kernel for Δ, μ is the Riesz mass of u, and ν is a nonnegative measure on the unit circle. Let ν_0 denote the total mass of ν, and define

$$v(z) = -\int_\Delta g(z, -|\zeta|)d\mu(\zeta) - \nu_0 P(z, \pi) + B(1).$$

Then v is s.h. in Δ, harmonic in $\Delta\backslash(-1, 0]$, with the same maximum on $\partial\Delta$ as u and the same Riesz mass in each disk $|z| \le R < 1$. Monotonicity properties of g and P show that for $r \in (0, 1)$

$$v(-r) = A(r, v) \le A(r, u) \le B(r, u) \le B(r, v) = v(r),$$

and for $|z| = r$

$$v(r) + v(-r) \le u(z) + u(-z).$$

Assume now that u is s.h. in Δ, that $\lambda \in (0, 1)$, that $B(1) = 1$, and that for all $r \in (0, 1)$,

$$(4)\qquad A(r, u) - (\cos \pi\lambda)B(r, u) \le 0.$$

The function

$$U(z) = \frac{2}{\pi}(\tan \frac{\pi\lambda}{2}) \operatorname{Re} \int_0^z \frac{t^{\lambda-1} - t^{1-\lambda}}{1 - t^2} dt$$

satisfies (4) with equality, and also has $U(e^{i\phi}) = 1$ for $|\phi| < \pi$. Applying (3) to v, and using the inequalities connecting u and v, one shows that

$$(5)\qquad B(r, u) \le U(r) \le C_1(\lambda) r^\lambda.$$

The case $\lambda = \frac{1}{2}$ is the "projection theorem" for harmonic measure, proved independently in 1933 by Beurling and R. Nevanlinna.

Let u be s.h. and nonconstant in $\mathbb{C}$, and suppose that the lim sup in (2) is finite. Subtracting an appropriate constant, we may assume that $A(r, u) \leq (\cos \pi\lambda)B(r, u)$ holds for all $r \in (0, \infty)$. Take $0 < r < R$, and apply (5) to the function $B(R)^{-1}u(Rz)$. The result is

$$r^{-\lambda}B(r) \leq R^{-\lambda}B(R)C_1(\lambda). \tag{6}$$

The left-hand side is positive for large r. Letting $R \to \infty$, we see that $\underline{\lim}_{R\to\infty} R^{-\lambda} \; B(R) > 0$. If this lim inf is $+\infty$, the limit α in the conclusion of Kjellberg's theorem exists and is infinite. If the lim inf is finite, then by letting $R \to \infty$ through a suitable sequence we see from (6) that the corresponding lim sup is also finite. Thus, u has order $\lambda < 1$. Let $\phi(r) = r^{-\lambda}B(r)$. A limiting case of (3) applied to the v corresponding to $u(zR)$ as $R \to \infty$ leads to a convolution inequality of the form

$$\phi(r) \leq \int_0^\infty \phi(t)k(rt^{-1})t^{-1}dt$$

for a certain positive function k. Essén proved a Tauberian theorem which asserts that for such k, if ϕ is bounded above on $(0, \infty)$ and $\underline{\lim}_{r\to\infty} \phi(r) > 0$, then $\lim_{r\to\infty} \phi(r)$ exists, thereby completing his proof of Kjellberg's theorem.

We discuss now some extremal problems of the cos $\pi\lambda$ type in which $A(r)$ and/or B(r) are replaced by other functionals measuring the growth of u. Let $m(r, u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u^+(re^{i\theta})d\theta$. Paley (1932) conjectured that if u is s.h. in $\mathbb{C}$ with order $\lambda \in [0, \infty)$ then

$$\underline{\lim}_{r\to\infty} \frac{B(r, u)}{m(r, u)} \leq \begin{cases} \pi\lambda \csc \pi\lambda, & \lambda \leq \frac{1}{2} \\ \pi\lambda, & \lambda > \frac{1}{2}. \end{cases}$$

This was proved by Govorov in 1969. The function $r^\lambda \cos \lambda\theta$ is extremal when $0 < \lambda \leq \frac{1}{2}$. For $\frac{1}{2} < \lambda < \infty$ an extremal is furnished by the "harmonic spline"

$$\begin{aligned} u(re^{i\theta}) &= r^\lambda \cos \lambda\theta, & |\theta| \leq \frac{\pi}{2\lambda}, \\ &= 0, & \frac{\pi}{2\lambda} \leq |\theta| \leq \pi. \end{aligned}$$

Govorov's theorem was extended to higher dimensions by B. Dahlberg (1972).

A. Edrei and W. Fuchs (1960) studied δ-s.h. functions of order $\lambda \in (0, 1)$. They proved a result known as the "ellipse theorem" which specifies the possible values of the "Nevanlinna deficiencies" $\delta(u)$ and $\delta(-u)$. Their analysis was based upon the fact that if a function v is harmonic in the slit disk $\Delta\backslash(-1, 0]$ then the new function

$$v_1(re^{i\theta}) = \int_{-\theta}^{\theta} v(re^{it})dt, \qquad 0 < \theta < \pi, \tag{7}$$

is harmonic in the upper half of Δ.

Edrei (1967) proved a sharp upper bound for the sum of the Nevanlinna deficiencies of a meromorphic function f in $\mathbb{C}$ of order $\lambda \in [0, \frac{1}{2}]$. He stated a conjecture for the case $\frac{1}{2} < \lambda < 1$, whose validity would follow from that of another, called the spread conjecture, giving precise lower bounds for the measure of sets $\{\theta : \log|f(re^{i\theta})| > 0\}$ in terms of the order and a deficiency of f, for certain special values of r. It turned out that Teichmüller (1939) had also made a conjecture of this type. The spread conjecture was proved by the reviewer (1973). Its proof was motivated by two techniques mentioned above : the idea of replacing a given s.h. function u by a new more symmetric one v obtained by sweeping the mass to the negative axis, and the harmonicity of the integral (7). Suppose that u is s.h. in Δ. Define a function u^* in the upper half Δ^+ of Δ by

$$u^*(re^{i\theta}) = \sup \int_E u(re^{it})dt,$$

where the sup is over all sets $E \subset [-\pi, \pi]$ of Lebesgue measure exactly 2θ.

Theorem. *u^* is s.h. in Δ^+.*

The spread conjecture can be proved by a harmonic majorization argument together with an extension of this theorem to the case of δ-subharmonic u. Extremals are given by harmonic splines with positive mass on the negative real axis and negative mass on the positive real axis. Using Edrei's notions of "Pólya peaks" and "local Phragmén-Lindelöf indicator," along with subharmonicity of the star-function, J. Rossi and A. Weitsman (1983) have shown how Paley's conjecture, the ellipse theorem, and the spread theorem, in more precise form, can all be proved by overlapping arguments.

Perhaps the leading unsolved problem in this complex is the "$k(\lambda)$ conjecture" of R. Nevanlinna (1929). In simplest form, it asks for the smallest constant $k(\lambda)$ for which

$$\varlimsup_{r\to\infty} \frac{\int_{-\pi}^{\pi} u^+(re^{i\theta})d\theta}{\int_{-\pi}^{\pi} u(re^{i\theta})d\theta} \le k(\lambda).$$

Here u is s.h. in $\mathbb{C}$ of order λ, and λ is positive real but not an integer. The traditional guess has been that an extremal should be furnished by our friend u_λ and its higher order brothers,

$$u_\lambda(re^{i\theta}) = (-1)^{[\lambda]} r^\lambda \cos\lambda\theta.$$

For $\lambda \in (0, 1)$ this is so, by a special case of the ellipse theorem. For $\lambda > 1$ it is open. The known methods all involve some type of symmetrization, and that theory is not yet sufficiently advanced to deal with problems when the supposed extremal, like u_λ for $\lambda > 1$, fails to be a symmetric decreasing function of θ.

The subharmonicity of the $*$-function has had application to the solution of other types of extremal problems. For example, in 1974 the reviewer used it to prove that the Koebe function has maximal integral means in the class of normalized univalent functions in the unit disk, and that integral means on circles of Green functions and harmonic measures of plane domains D increase when the domain is subjected to a circular symmetrization. Weitsman (1986) showed the same is true for solutions of certain nonlinear partial differential equations, including $u = -\log\rho$, where ρ denotes the Poincaré metric of D. Results of this sort involving "Schwarz symmetrization" in $\mathbb{R}^m$ can be found in the book [Ban]. A very general symmetrization theorem for p.d.e.'s was announced in [ALT].

A conjecture of the same vintage and character as Littlewood's $\cos\pi\lambda$ conjecture, but more difficult, was posed by Denjoy in 1907: Suppose that f is entire, and that $\log|f|$ has order $\lambda \in (0, \infty)$. Then f has at most 2λ "asymptotic values." The first proof was given by Ahlfors in his thesis (1930). Other proofs were published by Beurling and by Carleman in 1933. As traced in the survey [Bae], each of these proofs led to extensive subsequent developments. Those by Ahlfors and Beurling had a strongly geometric character and evolved into the theory of extremal length. Among the many descendants, we mention here only a theorem of S. Warschawski [1942] about conformal mapping of strip-like domains, which reverses the inequality in the result commonly known as Ahlfors's distortion theorem. Warschawski's theorem supplies a potent tool for the construction of examples. For instance, it can be used to show that R. Hornblower's (1971) condition $\int_0^1 \log B(r, u)\, dr < \infty$, which insures that a s.h. function u in the unit disk has asymptotic values on a dense subset of the boundary, is nearly sharp.

Carleman's method was more analytic. It involves convexity properties of the mean value $\frac{1}{2\pi}\int_{-\pi}^{\pi} u^2(re^{i\theta})d\theta$ when u is s.h. and nonnegative in a disk. Further work in this direction is due to, among others, Tsuji, Heins, and Hayman. Notable applications include exponential estimates for harmonic measure in terms of the function $\theta(r, D)$, which gives the angular measure of the largest arc of the intersection of the domain D with the circle $|z| = r$. These estimates play a role, for example, in theorems of Hayman and Weitsman (1975) about means and coefficients of "weakly univalent" analytic functions in the unit disk.

Subharmonic Functions, vol. 2 contains a full account of most of these results and methods, along with the solution of numerous nearby extremal problems and the associated regularity theory. And there is more. Much more. For instance, an exposition is given of various types of sparse sets in potential theory, with application to the theorems of Ahlfors-Heins and Hayman concerning exceptional sets for Phragmén- Lindelöf type theorems. The Wiman-Valiron theory about relations between the maximum modulus of an entire function and the size of the maximal term in its Taylor series is here, as is an extended form of Hall's lemma for estimating harmonic measure of sets in a half-plane, together with Hayman's counterexample to a natural conjecture for the sharp constant.

Subharmonic functions, vol. 2 provides updates to many of the topics treated in the books by Boas [Bo, 1954] and Tsuji [T, 1959]. It describes solutions to a number of the problems in classical function theory assembled by Hayman in the 1960s [H], and complements, with some small overlap, recently published books on potential and function theory by Duren [Du], Doob [Do], Garnett [G], and Koosis [K]. In its thoroughness it is reminiscent of Tsuji, but Hayman provides more motivation and, by being most generous with the details, becomes more easily digestible. The book is completely self-contained, except for material from *Subharmonic functions*, vol. 1 and introductory works on complex analysis such as Ahlfors's text. The bibliography is extensive. An especially helpful feature is the practice of indicating beside each item the pages in the text where it is referenced. In his preface to [G], Garnett states "Understand the figures and you understand the book." And indeed, his *Bounded analytic functions* is full of them. *Subharmonic functions*, vols. 1 and 2 together, contain 864 pages with, as best I can remember, no figures. The fact that both books succeed admirably in conveying related subjects illustrates the gratifying fact that in math, as in life, beauty can reveal itself in diverse guises.

Devotees of precise one-variable classical complex analysis will find *Subharmonic functions*, vol. 2 a gold mine. Because of the wealth of methods so clearly presented, many other analysts with no particular interest in min vs. max for subharmonic functions or the other major motifs will also find this a valuable book.

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