may arise. Finally, oscillations may develop. These last two must be controlled in a suitable manner or the result of the errant behavior must be incorporated into the final limit system of equations.

The first chapter is a review of basic notions with some refinements. Several methods of concentration are introduced, and the Young measure, useful in the study of oscillatory behavior, is described. The second chapter is a brief review of convexity. Quasiconvexity in the sense of Morrey, lower semicontinuity of variational integrals, and related issues are discussed in Chapter 3. Chapter 4 is devoted to concentration compactness with applications given to quasilinear elliptic systems and vorticity bounds for the Euler Equations. Chapter 5 concerns compensated compactness. The many applications include monotonicity methods and the div-curl lemma. Hyperbolic conservation laws are treated as well. Finally, Chapter 6 contains a discussion of the maximum principle for fully nonlinear equations and the homogenization of nondivergence structure equations. In explaining these contents, we have striven for extreme brevity. The ample citations and many historical references outline the contributions of many many scientists and offer avenues for further study.

The volume is dedicated to our beloved colleague Ron DiPerna.

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Finsler geometry and applications, by Aurel Bejancu. Ellis Horwood Limited, Chichester, 1990, 198 pp., $59.50. ISBN 0-13-317975-3

Let M denote an n-dimensional differentiable manifold with coordinate functions $\{x^j : j = 1, \ldots, n\}$ on a coordinate neighborhood U. The tangent vector of a smooth arc $\gamma : [t_0, t_1] \to U$, referred to a parameter t, has components $y^j = dx^j/dt$. If there is given a smooth nonnegative function F on the tangent bundle TM of M, an arc-length may be assigned to γ, namely

$$(1) \qquad s = \int_{t_0}^{t_1} F(x^1, \ldots, x^n; y^1, \ldots, y^n)\, dt.$$

In order to ensure that this integral be independent of the choice of the coordinates on U and of the parameter t, it is assumed that F is a scalar function on M that is homogeneous of the first degree in the directional arguments $\{y^j\}$. Under these circumstances F is called a metric function; for instance, if F^2 is an invariant positive definite quadratic form in $y^1, \ldots, y^n$, a Riemannian metric is thus defined. However, if F merely satisfies the aforementioned invariance conditions, it is said to give rise to a Finsler metric. It is remarkable that when Riemann introduced his quadratic metric he also suggested the possibility of other metrics, such as that defined by the fourth root of a homogeneous polynomial of order four [8, p. 278]. Thus it is fair to say that the concept of a Finsler metric had been anticipated by Riemann as early as 1854.

Nevertheless, the first significant developments concerning Finsler metrics emerged a half century later in the investigations of C. Carathéodory [3, 4] into discontinuous solutions in the calculus of variations of integrals such as (1). It was Carathéodory who introduced the idea of an indicatrix: this is the hypersurface

$$F(x^1, \ldots, x^n; y^1, \ldots, y^n) = 1$$

in the tangent space $T_p(M)$ of a point $p \in M$ with coordinates $\{x^j\}$. Clearly the indicatrix plays the role of a unit sphere; for a Riemannian metric it is an hyperellipsoid. The classical excess condition of Weierstrass for the integral (1) is tantamount to the convexity of the indicatrix, which clearly underscores the significance of this hypersurface. A systematic investigation of the local geometry of manifolds whose tangent spaces are endowed with such indicatrices was carried out in 1918 in the dissertation of P. Finsler [6]. Concepts such as angle, curvatures of curves and of surfaces were defined by Finsler in terms of a general metric function F, which led to generalizations of several basic theorems of the classical theory of surfaces.

In 1925 it was discovered independently by several investigators that a type $(0, 2)$ metric tensor can be defined in terms of the second derivatives of F^2 with respect to its directional arguments. For the case of a Riemannian metric function, this tensor reduces to the usual metric tensor. Thus, in this setting, the essential difference between Riemannian and Finsler geometry is reflected in the dependence of the metric tensor of the latter on directional arguments. This realization clearly dominated most subsequent developments whose obvious objective was the portrayal of Finsler geometry as a generalization of Riemannian geometry by means of the tensor calculus. This goal was pursued in a systematic manner primarily by L. Berwald [2], who used the Euler-Lagrange equations associated with the integral (1) to define connection coefficients and hence a notion of parallel displacement, together with a curvature tensor that allowed for the generalization of some local theorems of Riemannian geometry. However, a significant drawback of Berwald's connection is the fact that the length of a vector is not in general preserved under parallel displacement. This difficulty was overcome by the introduction of a new connection by E. Cartan [5], who also demonstrated that a complete theory of curvature for direction-dependent metrics requires three distinct types of curvature tensors. Although Cartan's monograph [5] dominated the subsequent literature more than any other single work, it did not entirely displace Berwald's theory: in fact, significant advances were achieved by a combination of both approaches, as is evidenced most strikingly by the posthumous papers of Berwald that appeared in the *Annals of Mathematics* (Princeton) during the 1940s. These developments stimulated further proposals of connections and corresponding theories of curvature of Finsler spaces and of their submanifolds. A fairly comprehensive description of this material, as it was known up to 1957, and related aspects thereof, is presented in the reviewer's book [9].

It has frequently been remarked that a characteristic feature of much of the literature on Finsler geometry is its dependence on the use of apparently unavoidable long and often quite intricate calculations based on the tensor calculus. No doubt this is one of several reasons why the subject has not at any time been regarded as one of the foremost central issues of modern mathematics.

On the other hand, Finsler metrics appear to provide, at least at first sight, an ideal background for a unified theory of electromagnetism and gravitation in a manner that would be analogous to the role played by Riemannian metrics in the relativistic theory of gravitation. While this objective proved to be a most powerful stimulus for the development of the subject, it also proved to be elusive; indeed, none of the numerous attempts to formulate such unified field theories has as yet given rise to a realistic physical theory that has gained some degree of acceptance.

In recent years, somewhat more modern treatments of Finsler geometry have been developed, particularly by M. Matsumoto and his collaborators, whose approach is based in the first instance on the notion of Finsler bundles and connections. In this context, one of the basic concepts is that of a nonlinear connection: this gives rise to a theory of torsion and curvature that is independent of Finsler metrics, the latter being introduced in an appropriate manner at a later stage. The invariants of the theories of Berwald, Cartan, and others emerge naturally within this setting, whose analytical complexities are again quite formidable. These developments are presented in Matsumoto's book [7], which also describes special Finsler spaces that may conceivably be of relevance to physical field theories. A different approach was suggested by the reviewer [10] in which the primary geometric objects are represented by a set of direction-dependent 1-forms, it being shown that thus a comprehensive theory, including the subsequent introduction of a metric, can be developed entirely without the imposition of the homogeneity conditions that are prevalent in all earlier treatments. As regards possible applications to physics, several investigators considered non-Abelian gauge theories in terms of some of the aforementioned geometrical structures. Much of this material is described in the fairly recent treatise by G. Asanov [1].

The book under review represents a further step in this direction. In order to indicate the extent to which the author's approach differs from earlier treatments, a brief summary is given of the concepts that are defined in Chapter 1. For a given manifold M, the vertical vector bundle VTM over the tangent bundle TM is defined as the subbundle of TTM whose fibres are the tangent spaces to the fibres of TM. A Finsler vector field on M is a section of VTM (and is therefore nothing other than a vector field on M with direction-dependent components). A Finsler differential 1-form is defined as a section of the vector space that is dual to VTM, with obvious extensions to Finsler tensor fields of type (p, q), including type $(0, 2)$ Finsler metrics. A nonlinear connection on TM is a distribution HTM in TTM that is complementary to VTM. Let ∇ denote a linear connection on VTM (defined as usual as a linear connection on a vector bundle). The pair $FC = (HTM, \nabla)$ is called a Finsler connection on M. An analysis of such connections gives rise to horizontal, mixed, and vertical curvature tensors. This formalism represents the basis of a detailed treatment in Chapter 2 of submanifolds. Chapter 3 is titled "Gauge theory on the tangent bundle." In essence this theory is concerned with variational principles for physical field theories that are represented by p components Q^A, which depend not only on the coordinates of M, but also on directional arguments. The action integral is defined on a compact domain of TM, and the Lagrangian is allowed to depend on the following: a Finsler metric on M, the fields Q^A, the derivatives of Q^A with respect to their directional arguments, and the co-

variant derivatives of Q^A defined by some nonlinear connection HTM. It is also assumed that the Lagrangian is invariant under the infinitesimal transformations induced on Q^A by a given Lie group G. Similarly, Chapter 4 is titled "Gauge theory on a fibre bundle" and presents an extension of the material of Chapter 3 in the sense that the role of TM is now played by an arbitrary fibre bundle.

The remaining chapters are devoted to applications of this formalism. Thus, in Chapter 5, field theories are considered for which the metric and the field variables depend on the coordinates of the total space of fibre bundle over M, the coordinates of the fibre being called perturbation parameters (no reasons for this terminology being given). This gives rise to formal generalizations of Maxwell's theory and of Yang-Mills theory. The formalism of Chapter 4 is applied in Chapter 6 to what is called multidimensional relativity theory: again the metric depends on the aforementioned perturbation parameters. In Chapter 7 some formal similarities between supermanifolds and the author's formalism are discussed, while the brief concluding Chapter 8 is concerned with the deformation of oriented media.

In summary: In the first part of the book, a very general and also quite complicated differential-geometric edifice is erected, which is then applied to the formulation of generalized physical field theories. This inevitably raises the question as to the ultimate purpose of such generalizations: is there some aspect of the immensely complicated field equations, or of (presumably as yet undiscovered) solutions thereof that renders them superior to the original equations? In almost all of the projected applications of Finsler geometry to physics questions of this kind are ignored. In this respect the book is no exception.

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