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Unbounded operator algebras and representation theory, by Konrad Schmüdgen. Birkhäuser and Akademie Verlag, 1990, 379 pp., $79.50. ISBN 3-7643-2321-3

The subject of the book has several beginnings: the Heisenberg commutation relations, quantum field theory, multivariable operator theory, representation theory for Lie algebras and Lie groups, and spectral decomposition theory, just to mention a few. An attractive feature of the subject of the book is the interplay between diverse tools from algebra, analysis, representation theory, and operator theory. The spectral theorem for a single self-adjoint operator shows that we may restrict attention to bounded operators by the use of the Cayley transform. Even if the operator under consideration is only symmetric with dense domain, but not self- adjoint, then the Cayley transform helps us to reduce the problem to an analysis of extensions of partial isometries. Recall the Cayley transform is a partial isometry. But if the index is nonzero (i.e., if the co-dimensional of the initial space and the final space are unequal) then there will not be self-adjoint extensions in the same given Hilbert space, but instead a dilated *extension Hilbert space* is needed for a better understanding of spectral resolutions and self-adjoint operator extensions. The analysis also dictates the study of two distinct commutants, a weak one and a strong commutant. This is (in quick review) the classical spectral multiplicity theory for a single symmetric operator in a given Hilbert space, and it dates back to John von Neumann, M. H. Stone, M. A. Naimark, and B. Sz.-Nagy. The present very interesting monograph is an attempt to carry over this analysis to *several variables* (**where unbounded operators cannot be avoided**), both the commutative case and the noncommutative one. Since the aim is to develop a useful spectral theory, the operator adjoint must be exploited, and attention is focused on representations which respect the operator adjoint: For algebras, we must study $*$-representations, i.e., the identity $\langle \pi(a)\varphi, \psi\rangle = \langle \varphi, \pi(a^*)\psi\rangle$ is assumed when π is the given representation, a is in the algebra, and $a \mapsto a^*$ is the given involution, $\langle\cdot, \cdot\rangle$ is the Hilbert space inner product, and, finally, the vectors φ, ψ are specified in a fixed dense domain $\mathscr{D}(\pi)$. Hence, it is understood that the operators $\pi(a)$ have the same dense domain $\mathscr{D}(\pi)$ for all a, and that $\pi(ab) = \pi(a)\pi(b)$ holds on $\mathscr{D}(\pi)$ for all a, b. In the Lie algebra case, such representations arise by the passing to the universal enveloping algebra. For elements x in the Lie algebra $\mathscr{G}$, we must then have $\langle \pi(x)\varphi, \varphi\rangle + \langle \varphi, \pi(x)\varphi\rangle = 0$ for all $\varphi \in \mathscr{D}(\pi)$. We consider $\mathscr{G}$ as a Lie algebra over $\mathbb{R}$, and define $x^* = -x$, $x \in \mathscr{G}$. This extends to the enveloping algebra, and we get a $*$-representation by virtue of the universal property of the enveloping algebra. Examples of such representations come from unitary representations of Lie groups G. Suppose U is such a representation, and that $\mathscr{G}$ is the corresponding Lie algebra with enveloping algebra $\mathscr{A}$. Then the derived infinitesimal representation $\pi = dU$ is a Hermitian representation of $\mathscr{A}$, and for the domain $\mathscr{D}(\pi)$ we may take the C^∞-vectors, or equivalently (by Dixmier-Malliavin) the Gårding vectors. The *integrability question* is the converse problem: To what extent is it possible to reconstruct a unitary representation from some given representation of the Lie algebra? Just as in the single operator case, we may try to write the given representation π of the Lie algebra in the form $\pi = dU$ for some unitary representation U. This cannot always be done. But it may be possible instead to have π **contained in** dU for U some unitary representation in the same Hilbert space, **or** in some **dilated** (or extended) Hilbert space. But even this weaker form of the exponentiation problem does *not* always have a solution. The book gives an authoritative exposition of this very interesting theory, and the author has himself made fundamental contributions to the subject.

It is also pointed out how the problem of exponentiating a representation of a Lie algebra may be thought of as an analogue to the problem from Wightman quantum field theory: "Reconstruct some quantum field from the corresponding moments of the field, i.e., the n-point functions, assuming that these are given for all n as distributions."

As can be expected, there are many technical questions which must first be studied and separated with clear examples. This is done very successfully and precisely by the author, and the book contains a wealth of illuminating examples, some connected to the pioneering paper by E. Nelson on analytic vectors (1959), and the by now familiar example of a pair of self-adjoint unbounded operators which commute on a core in the pointwise sense, but which do not have commuting spectral resolutions, i.e., the corresponding unitary one-parameter groups do not commute. This means that the corresponding two-dimensional abelian Lie algebra does *not* exponentiate to a unitary representation of $(\mathbb{R}^2, +)$. A fascinating feature of the book is an approach of the author to a class of such examples which surprisingly turn out to generate type III von Neumann algebra factors. The author also includes a clean, selfcontained, and illuminating exposition of the three techniques in the subject for reconstructing global representations from infinitesimal ones: (1) C^∞-vectors, (ii) analytic vectors, and (iii) positivity methods. Again, each one of the three approaches is illustrated with many interesting examples, some of which have not earlier appeared in the literature.

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Topics in Nevanlinna theory, by Serge Lang and William Cherry. Springer-Verlag, Berlin, Heidelberg, and New York, 1990, 174 pp., $18.00. ISBN 3-540-52785-0

It is well known that a nonconstant polynomial function $f: \mathbb{C} \to \mathbb{C}$ takes on every value in $\mathbb{C}$ the same number of times when multiplicities are taken into account. Nevanlinna theory, or value distribution theory, is a generalization of this fact to arbitrary nonconstant meromorphic functions. In this case, most values in $\mathbb{C}$ will occur infinitely many times, so one studies the number of values $f(z) = a$ with $z \in \mathbb{D}_r$ (the disc of radius r) asymptotically as $r \to \infty$.