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LIFTING OF COHOMOLOGY AND UNOBSTRUCTEDNESS OF CERTAIN HOLOMORPHIC MAPS

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Abstract. Let f be a holomorphic mapping between compact complex manifolds. We give a criterion for f to have *unobstructed deformations*, i.e. for the local moduli space of f to be smooth: this says, roughly speaking, that the group of infinitesimal deformations of f, when viewed as a functor, itself satisfies a natural lifting property with respect to infinitesimal deformations. This lifting property is satisfied e.g. whenever the group in question admits a 'topological' or Hodge-theoretic interpretation, and we give a number of examples, mainly involving Calabi-Yau manifolds, where that is the case.

One of the most important objects associated to a compact complex manifold X is its *versal deformation* or *Kuranishi family*

$$\pi\colon \mathcal{X} \to \mathrm{Def}(X);$$

this is a holomorphic mapping onto a germ of an analytic space $(\mathrm{Def}(X), 0)$ (the Kuranishi space) with the universal property that $\pi^{-1}(0) = X$ and that any sufficiently small deformation of X is induced by pullback from π by a map unique to 1st order. In general, $\mathrm{Def}(X)$ is singular and even nonreduced; in case $\mathrm{Def}(X)$ is smooth, i.e. a germ of the origin in $\mathbb{C}^N$, we say that X is *unobstructed*. In an analogous fashion, a holomorphic mapping

$$f\colon X \to Y$$

also possesses a versal deformation, which in this case is a diagram

$$\begin{array}{ccc} \tilde{f}\colon \mathcal{X} & \longrightarrow & \mathcal{Y} \\ & \searrow \quad \swarrow & \\ & \mathrm{Def}(f) & \end{array}$$

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with a similar universal property. Again we say that f is unobstructed if $\mathrm{Def}(f)$ is smooth.

Now in [R3], we gave a criterion which deduces the unobstructedness of a compact complex manifold X from a lifting property (in particular, deformation invariance) of certain cohomology groups associated to X; this implies in particular the unobstructedness of Calabi-Yau manifolds, i.e. Kähler manifolds with trivial canonical bundle K_X (theorem of Bogomolov-Tian-Todorov [B, Ti, To]), as well as that of certain manifolds with "big" anticanonical bundle $-K_X$. In this note we announce an extension of our criterion to the case of holomorphic maps of manifolds and discuss some applications, mainly to maps whose source is a Calabi-Yau manifold.

1. Generalities

Given a holomorphic map

$$f\colon X \to Y$$

of complex manifolds, we defined in [R1] certain groups T^i_f, $i \geq 0$, which are related to deformations of f; in particular, T^1_f is the group of 1st-order deformations of f. For our present purposes, it will be necessary to consider the corresponding relative groups $T^i_{\tilde f/S}$, which are associated to a diagram

$$\begin{array}{ccc} \tilde f\colon \mathcal{X} & \longrightarrow & \mathcal{Y} \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

with $\mathcal{X}/S$, $\mathcal{Y}/S$ smooth (we call such a map $\tilde f$ an S-map, or a deformation of f). In the notation of [R1, R2], we have

$$T^i_{\tilde f/S} = \mathrm{Ext}^i(\delta_1, \delta_0)$$

where $\delta_0\colon f^*\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$, $\delta_1\colon f^*\Omega_{\mathcal{Y}/S} \to \Omega_{\mathcal{X}/S}$ are the natural maps. As in [R1], we have an exact sequence

$$\begin{aligned} 0 \to T^0_{\tilde f/S} \to T^0_{\mathcal{X}/S} \oplus T^0_{\mathcal{Y}/S} \to \mathrm{Hom}_{\tilde f}(\Omega_{\mathcal{Y}/S}, \mathcal{O}_{\mathcal{X}}) \\ \to T^1_{\tilde f/S} \to T^1_{\mathcal{X}/S} \oplus T^1_{\mathcal{Y}/S} \to \mathrm{Ext}^1_{\tilde f}(\Omega_{\mathcal{Y}/S}, \mathcal{O}_{\mathcal{X}}) \to \cdots \end{aligned} \tag{1.1}$$

where $T^i_{\mathcal{X}/S} = H^i(T_{\mathcal{X}/S})$, $T_{\mathcal{X}/S}$ being the relative tangent bundle and similarly for $T^i_{\mathcal{Y}/S}$, $\mathrm{Hom}_{\tilde f}(\cdot,\cdot) = \mathrm{Hom}_{\mathcal{X}}(\tilde f^*\cdot,\cdot)$ and $\mathrm{Ext}^i_{\tilde f}(\cdot,\cdot)$ are its derived functors.

Now put $S_j = \mathrm{Spec}\,\mathbb{C}[\varepsilon]/(\varepsilon^j)$. Our main general result, which is an analogue for maps of a result given in [R3] for manifolds, is the following

Theorem-Construction 1.1. *Suppose given X_j/S_j, Y_j/S_j smooth and $f_j\colon X_j \to Y_j$ an S_j-map, for some $j \geq 2$, and let X_{j-1}/S_{j-1}, Y_{j-1}/S_{j-1}, $f_{j-1}\colon X_{j-1} \to Y_{j-1}$ be their respective restrictions via the natural inclusion $S_{j-1} \hookrightarrow S_j$. Then*

(i) *associated to f_j is a canonical element $\alpha_{j-1} \in T^1_{f_{j-1}/S_{j-1}}$;*

(ii) *given any element $\alpha_j \in T^1_{f_j/S_j}$ which maps to α_{j-1} under the natural restriction map $T^1_{f_j/S_j} \to T^1_{f_{j-1}/S_{j-1}}$, there are canonically associated to α_j deformations X_{j+1}/S_{j+1}, Y_{j+1}/S_{j+1} and an S_{j+1}-map $f_{j+1}\colon X_{j+1} \to Y_{j+1}$, extending X_j/S_j, Y_j/S_j and $f_j\colon X_j \to Y_j$ respectively.*

The proof is analogous to that of Theorem 1 in [R3] and will be presented elsewhere. In view of this theorem it makes sense to give the following

Definition 1.2. A map $f\colon X \to Y$ is said to satisfy the T^1-lifting property if for any deformation $f_j\colon X_j/S_j \to Y_j/S_j$ of f and its restriction f_{j-1}: $X_{j-1}/S_{j-1} \to Y_{j-1}/S_{j-1}$, the natural map

$$T^1_{f_j/S_j} \to T^1_{f_{j-1}/S_{j-1}}$$

is surjective.

Abusing terminology somewhat, we will say that T^1_f is *deformation-invariant* if the groups $T^1_{f_j/S_j}$ are always free S_j-modules and their formation commutes with base-change. Note, trivially, that whenever T^1_f is deformation-invariant, f satisfies the T^1-lifting property. As an easy consequence of Theorem 1.1, we have the following

Criterion 1.3. *Suppose $f\colon X \to Y$ is a map of compact complex manifolds satisfying the T^1-lifting property* (*e.g. T^1_f is deformation-invariant*); *then f is unobstructed.*

Remark 1.4. Various variants of this criterion are possible, e.g. for deformations of maps $f\colon X \to Y$ with fixed target Y. In the special case that f is an embedding, with normal bundle N, we obtain that the Hilbert scheme of submanifolds of Y is smooth at the point corresponding to $f(X)$ provided $H^0(N)$ satisfies the lifting property (e.g. is deformation-invariant). Also, the *converse* to Criterion 1.3 is trivially true, though we shall not need this.

2. Applications

Unless otherwise specified, all spaces X, Y considered here are assumed smooth.

Theorem 2.1. *Let X be a Calabi-Yau manifold and $f\colon Y \hookrightarrow X$ the inclusion of a smooth divisor. Then f is unobstructed and moreover the image and fibre of the natural map* $\mathrm{Def}(f) \to \mathrm{Def}(X)$ *are smooth.*

Proof. In this case we may identify T^1_f with $H^1(T')$ where T' is defined by the exact sequence

$$0 \to T' \to T_X \to N_{Y/X} \to 0, \tag{2.1}$$

and it will suffice to prove deformation invariance of $H^1(T')$. Now identifying $T_X \cong \Omega^{n-1}_X$, $N_{Y/X} \cong \Omega^{n-1}_Y$, $n = \dim X$, we may write the cohomology sequence of (2.1) as

$$0 \to H^{n-1,0}(Y) \to H^1(T') \to H^{n-1,1}(X) \xrightarrow{f^*} H^{n-1,1}(Y) \cdots .$$

As $H^{n-1,0}(Y)$ and $\ker(f^*)$ are both deformation-invariant, so is $H^1(T')$, hence f is unobstructed, and since moreover the former groups are the respective tangent spaces to the fibre and image of $\mathrm{Def}(f) \to \mathrm{Def}(X)$, the latter are smooth. Q.E.D.

A similar argument can be used to reprove a recent theorem of C. Voisin [V] (see *op. cit.* for examples and further results):

Theorem 2.2 (Voisin). *Let X be a Kähler symplectic manifold, with (everywhere nondegenerate) symplectic form $\omega \in H^0(\Omega_X^2)$, and $f: Y \to X$ a Lagrangian embedding, i.e. $f^*\omega = 0$ and $\dim Y = \frac{1}{2}\dim X$. Then f is unobstructed and the image and fibre of the natural map $\mathrm{Def}(f) \to \mathrm{Def}(X)$ are smooth.*

Proof. In this case we may identify $T_X \cong \Omega_X$, $N_{Y/X} \cong \Omega_Y$, and we may argue as in the proof of Theorem 2.1 (note that this property of being Lagrangian is *open*).

Next we consider deformations of fibre spaces $f: X^n \to Y^m$ with X Calabi-Yau (i.e. f is a flat map whose fibres are reduced and connected). Note that for a fibre space f, its general fibre is clearly a Calabi-Yau manifold. Also, it follows easily from the sequence (1.1) that $\mathrm{Def}(f) \hookrightarrow \mathrm{Def}(X)$. When $R^1 f_* \mathcal{O}_X = 0$, the morphism $\mathrm{Def}(f) \to \mathrm{Def}(X)$ is an isomorphism by a theorem of Horikawa [H], hence in that case unobstructedness of f follows from that of X. We will consider here two extreme cases: namely $m = n - 1$ and $m = 1$.

Theorem 2.3. *Let $f: X \to Y$ be an elliptic fibre space (i.e. general fibre elliptic curve) with X Calabi-Yau. Then f is unobstructed.*

Proof. Using the usual exact sequence (1.1) and Criterion 1.3, it suffices to prove the deformation invariance of

$$\ker(H^1(T_X) \xrightarrow{\alpha} H^0(Y, R^1 f_* \mathcal{O}_X \otimes T_Y)).$$

Now by relative duality we have

$$R^1 f_* \mathcal{O}_X \cong \omega_{X/Y}^{-1} \cong \omega_Y,$$

hence we may identify α with the push-forward map (or "integration over the fibre")

$$H^{n-1,1}(X) \to H^{n-2,0}(Y),$$

and in particular $\ker \alpha$ is deformation-invariant. (Note that we have $\mathrm{Def}(f) \cong \mathrm{Def}(X)$ whenever $\alpha = 0$, e.g. $H^{n-2,0}(Y) = 0$, which holds whenever $H^{n-2,0}(X) = 0$.)

Theorem 2.4. *Let $f: X \to C$ be a fibre space from a Calabi-Yau manifold to a smooth curve. Then f is unobstructed.*

Proof. Note that for any fibre Y of f we have

$$h^0(\mathcal{O}_Y(Y)) = h^0(\mathcal{O}_Y) = 1,$$

and it follows that the scheme $\mathrm{Div}^0(X)$ parametrizing reduced connected effective divisors of X is smooth and 1-dimensional locally at the point corresponding to Y. Consequently if we denote by

$$p: Z \to \mathrm{Div}^0(X)$$

the universal family and $q: Z \to X$ the natural map, then we have in fact a 1-1 correspondence between morphisms $f: X \to C$ as above and smooth compact connected 1-dimensional components $C \subset \mathrm{Div}^0(X)$ such that $q|p^{-1}(C)$ is an isomorphism. Now it follows from Theorem 2.1 and its proof that for any smooth fibre Y of f, the locus $D' \subset \mathrm{Def}(X)$ of deformations over which Y extends is smooth and *independent* of Y. It follows that almost all, hence all, of C as component of $\mathrm{Div}^0(X)$ in fact extends over D', hence so does f, so that $D' = \mathrm{Def}(f)$, proving the theorem.

In the intermediate cases, we have only much weaker results:

Theorem 2.5. *Let $f\colon X \to Y$ be a smooth morphism and assume either*
(i) K_X *is trivial; or*
(ii) $K_{X/Y}$ *is trivial.*
Then $\mathrm{Def}(f) \to \mathrm{Def}(Y)$ *has smooth fibres.*

Proof. We will prove (ii), as (i) is similar. It suffices to prove the deformation invariance of $H^1(T_{X/Y})$, where $T_{X/Y}$ is the relative (vertical) tangent bundle. Now we have

$$T_{X/Y} \cong \Omega^{n-1}_{X/Y} \otimes K^{-1}_{X/Y} \cong \Omega^{n-1}_{X/Y} \qquad n = \dim(X/Y)\,.$$

By relative Hodge theory, $H^1(\Omega^{n-1}_{X/Y})$ is a direct summand of $H^n(f^{-1}\mathcal{O}_Y)$, and it will suffice to prove the deformation invariance of the latter. We have a Leray spectral sequence

$$H^p(Y, R^q f_* f^{-1}\mathcal{O}_Y) \Rightarrow H^n(f^{-1}\mathcal{O}_Y)\,. \tag{2.2}$$

However $H^p(Y, R^q f_* f^{-1}\mathcal{O}_Y) = H^{p,0}(Y, R^q f_* \mathbb{C}_X)$ is a direct summand of $H^p(Y, R^q f_* \mathbb{C}_X)$, hence the degeneration of the Leray spectral sequence of $\mathbb{C}_X$ implies that of (2.2), hence the deformation invariance of $H^n(f^{-1}\mathcal{O}_Y)$.

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Added in proof

The above ideas are pursued further in the author's preprints, *Hodge theory and the Hilbert scheme* (September 1990) and *Hodge theory and deformations of maps* (January 1991).

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