APPEARED IN BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 26, Number 1, Jan 1992, Pages 153-157

BOOK REVIEW

Topics in Nevanlinna theory, by Serge Lang and William Cherry. Springer-Verlag, Berlin, Heidelberg, and New York, 1990, 174 pp., \$18.00. ISBN 3-540-52785-0

It is well known that a nonconstant polynomial function $f\colon \mathbb{C} \to \mathbb{C}$ takes on every value in $\mathbb{C}$ the same number of times when multiplicities are taken into account. Nevanlinna theory, or value distribution theory, is a generalization of this fact to arbitrary nonconstant meromorphic functions. In this case, most values in $\mathbb{C}$ will occur infinitely many times, so one studies the number of values $f(z) = a$ with $z \in \mathbb{D}_r$ (the disc of radius r) asymptotically as $r \to \infty$.

More precisely, we make the following definitions. Fix $a \in \mathbb{P}^1(\mathbb{C}) := \mathbb{C} \cup \{\infty\}$, and assume for simplicity that $f(0) \neq a$. Let the *counting function* $n_f(r)$ be defined as the number of poles of f in $\mathbb{D}_r$, and let

$$n_f(a,r) = \begin{cases} n_{1/(f-a)}(r), & a \in \mathbb{C}; \\ n_f(r), & a = \infty. \end{cases}$$

Then let

$$N_f(a,r) = \int_0^r n_f(a,r)\,\frac{dr}{r},$$

so that solutions $f(z) = a$ are counted with weights $\log(r/|z|)$. This turns out to be the most useful definition.

The main theorems of Nevanlinna theory establish asymptotic upper and lower bounds on $N_f(a,r)$; they are named (appropriately) the First Main Theorem and the Second Main Theorem, respectively.

Before stating these results, however, a few more definitions are necessary. First, we define the *chordal distance* on $\mathbb{P}^1(\mathbb{C})$ (as on page 18) by letting

$$\|w,w'\|^2 = \frac{|w-w'|^2}{(1+|w|^2)(1+|w'|^2)}$$

for $w, w' \in \mathbb{C}$ and

$$\|w,\infty\|^2 = \|\infty,w\|^2 = \frac{1}{1+|w|^2}.$$

Then the *proximity function* is defined by the formula

$$m_f(a,r) = \int_0^{2\pi} -\log\|f(re^{i\theta}),a\|\,\frac{d\theta}{2\pi}.$$

Finally, let the *height function*, or *characteristic function*, of f be

$$T_f(a,r) = m_f(a,r) + N_f(a,r) + \log \|f(0),a\|.$$

One way of stating the First Main Theorem, then, is that $T_f(a,r)$ is independent of $a \neq f(0)$; hence we may shorten the notation to $T_f(r)$. This theorem is also written

$$m_f(a,r) + N_f(a,r) = T_f(r) + O(1). \tag{1}$$

Since the chordal distance is always ≤ 1, we have $m_f(a,r) \geq 0$; hence

$$N_f(a,r) \leq T_f(r) + O(1)$$

is the desired upper bound on $N_f(a,r)$. This corollary is called the *Nevanlinna inequality*.

The question of producing lower bounds for $N_f(a,r)$ is quite a bit more delicate, due in part to functions such as e^z, for which $N(0,r) = N(\infty,r) = 0$. Therefore, we need to take several values of a into account: let $a_1,\ldots,a_q$ be distinct points in $\mathbb{P}^1(\mathbb{C})$; then the Second Main Theorem asserts that

$$\sum_{i=1}^{q} N_f(a_i,r) \geq (q-2)T_f(r) + N_{f,\mathrm{Ram}}(r) - O_{\mathrm{exc}}(\log rT_f(r)).$$

Here the notation $\alpha = O_{\mathrm{exc}}(\beta)$ means that there exists a constant $c > 0$ and a subset E of the positive real line with finite Lebesgue measure, such that $|\alpha(r)| \leq c\beta(r)$ for all $r \notin E$. Also, $N_{f,\mathrm{Ram}}(r)$ is a weighted (by $\log(r/|z|^2)$) count of the ramification points of f.

Via (1), the inequality can also be written

$$S := \sum_{i=1}^{q} m_f(a_i,r) - 2T_f(r) + N_{f,\mathrm{Ram}}(r) \leq O_{\mathrm{exc}}(\log rT_f(r)). \tag{2}$$

In *Topics in Nevanlinna theory,* Lang and Cherry derive a version of the Second Main Theorem with a sharper error term; a special case would be

$$S \leq (1+\epsilon)\log T_f(r) + O_{\mathrm{exc}}(1).$$

The theory of the error term is much richer than the above stark estimate indicates, though. To describe it further, let us digress briefly into number theory. A famous theorem of Roth is the following.

Theorem (Roth, [R]). *Fix an algebraic number α and fix $\epsilon > 0$. Then for all but finitely many rational numbers m/n (written in lowest terms),*

$$\left|\frac{m}{n} - \alpha\right| > \frac{1}{|n|^{2+\epsilon}}. \tag{3}$$

C. Osgood [O] was the first to observe a similarity between the roles of the constant 2 in Roth's theorem and in Nevanlinna's Second Main Theorem. This analogy, which is purely formal at this time, has been described more explicitly in [V]. In particular, let $a_1, \dots, a_q$ be the conjugates of α over $\mathbb{Q}$, and let

$$m\left(a, \frac{m}{n}\right) = -\log\left\|\frac{m}{n} - a\right\|;$$
$$T\left(\frac{m}{n}\right) = \log|n|.$$

Then (3) can be written

$$S := \sum_{i=1}^{q} m(a_i, x) - 2T(x) \le \epsilon\, T(x) + O_{\text{exc}}(1), \tag{4}$$

where x ranges over all rational numbers and the exceptional set in O_{exc} is now taken to be finite. (Here we added the conjugates of α not only to enhance the similarity to (2), but also to make the sum $\sum_i m(a_i, r)$ defined over $\mathbb{Q}$, which is necessarily for certain technical reasons.) Note that, in number theory, however, the emphasis is on finding upper bounds for $m(a, r)$ rather than lower bounds for $N(a, r)$.

In the 1960s, Lang conjectured that (4) could be sharpened considerably, to

$$S \le (1 + \epsilon)\log T(x)$$

or even that, if ψ is a positive increasing function satisfying

$$\sum_{i=1}^{\infty} \frac{1}{i\psi(i)} < \infty,$$

then

$$S \le \log \psi(T(x)).$$

It was this conjecture that motivated Lang's work leading up to this book.

In the case of Nevanlinna theory, assume ψ is a positive increasing function satisfying

$$b_0(\psi) := \int_e^\infty \frac{dx}{x\psi(x)} < \infty,$$

and let

$$S(F, c, \psi, r) = \log F(r) + \log\bigl\{\psi(F(r))\psi(crF(r)\psi(F(r)))\bigr\}.$$

Then Lang and Cherry prove in their book that the Second Main Theorem can be refined to a proof that in (2),

$$S \le \tfrac{1}{2} S(B_q T_f^2, b_1, \psi, r) + b \tag{5}$$

for all $r \ge r_1$, outside some set of Lebesgue measure $\le 2b_0(\psi)$, where B_q, r_1, b, and b_1 are suitable constants.

As one might guess, over the years Nevanlinna theory has been generalized in several directions, two of which are relevant to *Topics in Nevanlinna theory.* First of all, Stoll [S] and Carlson and Griffiths [CG] generalized the First and Second Main Theorems to the case of $f\colon \mathbb{C}^n \to V$, where V is a compact complex projective manifold of dimension n and the jacobian determinant of f is not identically zero. The definitions and theorems are much the same; it is only necessary to replace the targets $a \in \mathbb{P}^1(\mathbb{C})$ with divisors D on V with bounds on the types of singularities on D (simple normal crossings). Here Lang obtains the same sort of bounds as above, except that one needs also to take into account the complexity of the singularities of the divisor.

In another direction, Griffiths and King [GK] generalized Nevanlinna theory to consider maps $f\colon Y \to \mathbb{C}$, where $p\colon Y \to \mathbb{C}$ is a finite branched holomorphic covering of $\mathbb{C}$. In this case we count solutions of $f(y) = a$ in the set $p^{-1}(\mathbb{D}_r)$. Again, an error term of the form (5) holds; this is Cherry's contribution to the book. Here, however, an additional term $-N_{p,\mathrm{Ram}}(r)$ appears in the definition of S on the left, and the degree of p must appear not only as a factor in front of $S(F, c, \psi, r)$ on the right, but also either in one of its arguments or in the size of the exceptional set.

Then, of course, one could superimpose these two generalizations, and prove the main theorems for a finite cover of $\mathbb{C}^n$. This case is treated also by Cherry.

All told, then, there are four contexts in which one can prove the main theorems; each is proved in one of the book's four chapters. Each chapter gives a largely self-contained proof of both main theorems in its particular context. While this may possibly be a bit redundant, it has the advantage of offering a complete exposition of the classical case ($f\colon \mathbb{C} \to \mathbb{C}$), while still giving very thorough and readable accounts of the more general cases without hand-waving about a proof being exactly the same as before, only different.

The book is remarkably free of errors, although a few do occur. Some of the more serious are on the bottom of page 24, where a minus sign should be an equals sign, and on page 164, where the last exponent in $\delta(Y/\mathbb{C}^n, k)$ should be $n/(n + k)$ Also, in Part Two, Cherry refers repeatedly to "Stoke's Theorem."

Moreover, since certain notations, such as $S(F, b_1, \psi, r)$ and the chordal distance $\|\cdot,\cdot\|$, occur repeatedly throughout the book, an index of notations would be a welcome addition. The subject index is also a bit sparse—it omits the reference to Khintchine on page 41, for example.

Overall, though, the book is very clear and well written. For the experts, it gives a thorough account, *ab ovo,* of the newly discovered structure of the error term. Yet the extra care given to the error term does not clutter up the proof of the Second Main Theorem, so it serves very well also as an introduction to the subject for the more general audience.

References

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