Overall, though, the book is very clear and well written. For the experts, it gives a thorough account, *ab ovo,* of the newly discovered structure of the error term. Yet the extra care given to the error term does not clutter up the proof of the Second Main Theorem, so it serves very well also as an introduction to the subject for the more general audience.

REFERENCES

[CG] J. Carlson and P. Griffiths, *A defect relation for equidimensional holomorphic mappings between algebraic varieties*, Ann. Math. **95** (1972), 557–584.

[GK] P. Griffiths and J. King, *Nevanlinna theory and holomorphic mappings between algebraic varieties*, Acta Math. **130** (1973), 145–220.

[O] C. Osgood, *A number theoretic-differential equations approach to generalizing Nevanlinna theory*, Indian J. of Math. **23** (1981), 1–15.

[R] K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika **2** (1955), 1–20.

[S] W. Stoll, *The Ahlfors-Weyl theory of meromorphic maps on parabolic manifolds*, Lecture Notes in Math., vol. 981, Springer-Verlag, New York, 1981.

[V] P. Vojta, *Diophantine approximations and value distribution theory*, Lecture Notes in Math., vol. 1239, Springer-Verlag, New York, 1987.

PAUL VOJTA
UNIVERSITY OF CALIFORNIA, BERKELEY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 26, Number 1, January 1992

0273-0979/92 $1.00 + $.25 per page

Set-valued analysis, by Jean-Pierre Aubin and Helene Frankowska. System & Control, Foundation & Applications, vol. 2, Birkhauser, Boston, 1990, xix + 462 pp., $65.00. ISBN 0-8176-3478-9

A set-valued function is nothing more than a function from a set into a power set of another set. Set-valued functions come in many shapes and sizes and have many names: relation, carrier, correspondence, multifunction, point-to-set map, to name a few. *Set-valued analysis*, not surprisingly, was born with point-set topology and is associated with the names of Painlevé, Bouligand, Hausdorff, and above all Kuratowski. As the present authors point out

> Set-valued maps were abandoned by the authors of *Bourbaki's topologie générale* who chose to restrict their study to single-valued maps, regarding set-valued maps as single-valued maps from a set to the power set of another set of factorizing single-valued maps to make them bijective.

The authors then make the case for the inappropriateness of this approach both practically and theoretically. I will illustrate with two theoretical examples. Let us fix sets X and Y and $\Omega: X \to 2^Y$. Since typically, Y is well structured (a metric space, normed space, topological space) it makes sense to try to preserve this structure by viewing $\Omega(x)$ as a set in Y. The change of perspective has fruitful consequences. Functions can be identified with set-valued functions with singleton images; the inverse $\Omega^{-1}: Y \to 2^X$ is defined by $x \in \Omega^{-1}(y)$ if

$y \in \Omega(x)$ and the first bane of functions disappears, there is no question that Ω^{-1} is "well defined;" intuition is maintained and properties of Y are not forced into hyperproperties of 2^Y. Properties of Ω or Ω^{-1} are identifiable with properties of the graph $\mathrm{Gr}(\Omega) := \{(x, y): y \in \Omega(x)\}$. Thus Ω is said to be *convex* or *closed* when $\mathrm{Gr}(\Omega)$ is a convex set or a closed set in $X \times Y$. When $\mathrm{Gr}(\Omega)$ is a convex cone, Ω is called a *convex process.* Such set-valued processes arise naturally in mathematical economics and in control theory. Most of the basic theory of linear operators has a complete analogue for convex processes. One also pays attention to the (effective) *domain* $D(\Omega) := \{x: \Omega(x) \neq \varnothing\}$ and to the *range* $R(\Omega) := \{y: y \in \Omega(x), \exists x \in X\}$, and the range of a correspondence is the domain of its inverse.

Of course, many objects in many parts of mathematics are naturally multivalued. Consider the antiderivative, the complex square root, branches of analytic functions, the solution to an undetermined system of equations, a linear programming feasible set, and much of mathematical economics.

For this reviewer, the classical Open Mapping/Closed Graph Theorem makes a splendid example of what is gained by reducing dependence on functions. Consider Banach spaces X and Y. One says that $\Omega: X \to 2^Y$ is *lower semicontinuous* (LSC) at x_0 in X if whenever $N(y_0)$ is a neighbourhood of $y_0 \in \Omega(x_0)$ there exists a neighbourhood $U(x_0)$ such that $\Omega(x) \cap N(y_0) \neq \varnothing$ for x in $U(x_0)$. This is one of the most fruitful ways of generalizing the continuity property of a function. One says that $\Omega: X \to 2^Y$ is open at y_0 in Y if whenever $N(x_0)$ is a neighbourhood of $x_0 \in \Omega^{-1}(y_0)$ then $y_0 \in \mathrm{int}[\Omega(N(x_0))]$. An easy computation shows that Ω is LSC at x_0 precisely if Ω^{-1} is open at x_0.

The Closed Graph Theorem (due to various authors and provable much as the classical version [H]) now becomes:

Theorem 1 (Closed Graph). *Let $\Omega: X \to 2^Y$ be closed and convex. Then Ω is LSC at any point interior to the domain of Ω.*

The Open Mapping Theorem becomes:

Theorem 2 (Open Mapping). *Let $\Omega: X \to 2^Y$ be closed and convex. Then Ω is open at any point interior to the range of Ω.*

Indeed Theorems 1 and 2 are entirely dual to each other: one merely reverses the roles of X and Y and of Ω and Ω^{-1}. There is no work needed to deduce one from the other through factorization as in most classical proofs. Some of this is discussed by Aubin and Frankowska. Of course, there are many fine applications of these theorems other than to closed linear operators.

Often set-valued functions arise naturally: as with maximal monotone operators [ET], convex analysis [R], or the Clarke generalized derivative [C]. Sometimes they are merely technical devices. For instance, it is often easy to define a LSC multifunction with prescribed properties, but hard to see that there is continuous single-valued way of doing so. One is lead to ask about "selections:" A lovely exemplar is proved in [H] and in the book under review.

Theorem 3 (Michael Selection). *Suppose X is a metric space* (*paracompact does*) *and Y is a Banach space.*

Let $\Omega: X \to 2^Y$ be LSC with closed nonempty convex images. Then there exists a continuous mapping $\sigma: X \to Y$ with $\sigma(x) \in \Omega(x)$ for all x in X.

As an application, suppose that $f: X \to R$ is lower semicontinuous (as a function not as a multifunction!) and suppose that $g: X \to R$ is upper semicontinuous. Suppose that $f(x) \geq g(x)$ for all x in X. Then $\Omega(x) := \{r: g(x) \leq r \leq f(x)\}$ satisfies the hypotheses of the Michael Selection Theorem. In consequence one can "sandwich" a continuous $h: X \to R$ with $g(x) \leq h(x) \leq f(x)$ for all x in X. This is a version of a celebrated interpolation result due variously to Dowker, Hahn, and Katetov.

Similar remarks apply to questions about measurable selections, inverse and implicit function theorems, derivative multifunctions, variational inequalities, differential inclusions, and the like. The authors of the book under review have been at the forefront of the development and application of set analysis techniques in control theory, nonlinear analysis, differential equations, and other fields. This book is an important part of a general venture to reestablish *set-valued analysis* as a basic mode of mathematical thought.

Set-valued analysis has ten chapters and a substantial bibliography arranged as follows:

Chapter 1 contains a basic primer on continuity of set-valued maps. Chapter 2 looks at the Closed Graph and Open Mapping Theorems and related topics for convex processes. Chapter 3 considers equilibria from the perspective of Ky Fan's inequality and at inverse function results based in part on Ekeland's variational principle. Chapter 4 opens the door on tangency considered, as is natural, via set-valued maps. In this case the set-valued map $\Omega(x) = T(C, x)$ represents some tangential approximation to a set C in a Banach space. Chapter 5 builds on this to examine derivatives and higher order derivatives for set valued maps: in essence by considering the tangent cone to the graph of Ω as the graph of a "derivative" to Ω. Chapter 6 undertakes a similar task for an extended real-valued function f by using the *epigraph*, $\text{epi}(f) = \{(x, r): f(x) \leq r\}$. This has been hugely successful for convex functions [R] and quite successful for Lipschitz functions [C]. Chapter 7 then returns to convergence of sequences of graphs and epigraphs. This is an area of growing importance as present researchers in nonlinear analysis become convinced that pointwise limits are inappropriate in many cases: say in the study of second-order behaviour of a function. Chapter 8 studies the beautiful field of measurability and integration of set-valued functions. It includes a discussion of measurable selections, of convexity of the integral due to Aumann, Debreu, Olech, and others and of the "bang-bang" principle. In Chapter 9 many lovely selection theorems are presented: Michael's Theorem for LSC maps, the approximate selection theorem for upper semicontinuous maps, and of the existence of selections with special properties such as Lipschitzness. Finally, Chapter 10 makes a brief survey of the expanding field of *differential inclusions* which studies when and how can one solve $x'(t) \in \Omega(t, x(t))$.

This book will be a valuable resource for workers in many fields. It contains a lot of information and many interesting asides. It is clearly written. It is particularly good on areas such as selections in which the general topography is well understood and some common understanding has been reached. Some of these topics are well covered in [K-T]. In areas such as epi-differentiation this communality is yet to be achieved and the book suffers accordingly. The theorems in these chapters come fast and furious with little texture for the uninitiate. I would personally have enjoyed a deeper and more leisurely voyage with fewer stops en route.

Thus, I recommend this book as one to dig into with considerable pleasure when one already knows the subject rather than a book from which to start to learn it. In conclusion, however, *Set-valued analysis* goes a long way toward providing a much needed basic resource on the subject.

References

[C] F. H. Clarke, *Optimization and nonsmooth analysis*, Wiley, New York, 1983.

[E-T] I. Ekeland and R. Temam, *Convex analysis and variational problems*, North-Holland, Amsterdam, 1976.

[H] R. B. Holmes, *Geometric functional analysis and applications*, Springer-Verlag, New York, 1975.

[K-T] E. Klein and A. C. Thompson, *Theory of correspondences*, Wiley, New York, 1984.

[R] R. T. Rockafellar, *Conjugate duality and optimization*, SIAM Publications, Philadelphia, 1974.

Jonathan Borwein
University of Waterloo

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 26, Number 1, January 1992

0273-0979/92 $1.00 + $.25 per page

Large deviation techniques in decision, simulation, and estimation, by James A. Bucklew. John Wiley & Sons, 1990, 270 pp., $49.95. ISBN 0-471-61856-X

1. Deficiencies and virtues

In more ways than one, Professor Bucklew's book is a large deviation among all probability texts I have ever seen. The book has two goals: to present the basic results of a rapidly expanding branch of probability known as the theory of large deviations and to apply the theory to a number of problems arising in electrical engineering. Unlike all other texts on large deviations that have been published to date, this one is addressed to the engineering community, not to mathematicians. With this audience in mind, the author remarks in the preface that the theory of large deviations has a great potential for applications but is "unusually technical." His intention is to expose this material mostly by means of heuristic arguments while at the same time presenting "the main ideas and possible groundwork for the full-blown treatment." Approximately, half the book is devoted to applications. Given the audience to whom the book is addressed, it certainly sounds like an excellent plan.

Professor Bucklew has set himself a formidable task. It is one thing to write a technically correct book that has precise statements and complete proofs of all the results. Most books in mathematics are of this type. It is another thing—perhaps even more demanding—to write a book that omits complete proofs but motivates all the results by clear heuristic arguments.[1] Composing successfully

[1] Mark Kac was one of the masters of this type of exposition.