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Handbook of differential equations, by Daniel Zwillinger. Academic Press, Inc., Boston, MA, 1989, xx+ 673 pp., \$49.95. ISBN 0-12-784390-6

Reviewers for this journal are invited to give a somewhat discursive account, and my review will not deviate from the custom so established. I taught undergraduate differential equations for the first time at Harvard University in 1948 and have taught it more or less regularly ever since. Added to that are about a dozen graduate courses on differential inequalities here and abroad. A few thoughts arising from this experience are summarized first. Next comes a survey of the current scene as reflected by popular texts, and finally Zwillinger's book.

For those who, like me, prefer to start a detective story by peeking at the end, I can save time by saying that Zwillinger's book is an outstanding scholarly achievement, original in both plan and execution. It will surely appeal to all who have more than a passing interest in differential equations. If you have read this far, you probably qualify; so you might as well skip the review and get the book.

GENERAL REMARKS

One of the charms of elementary differential equations is that most of the proofs are easy, yet they yield results of obvious importance. For example, to establish uniqueness for the solutions of

$$y'' + p(t)y' + q(t)y = f(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_1$$

on an interval containing t_0, let $w = u^2+(u')^2$ where u is the difference of two solutions. If the coefficients p and q are bounded, w satisfies a differential inequality that implies $w = 0$ and uniqueness follows. A similar method gives uniqueness for both linear and nonlinear equations of arbitrary order.

The importance of uniqueness can hardly be overestimated. If a differential equation arises from a physical system, it is only by uniqueness that we can say that our solution represents *the behavior of the system*, not merely *a possible behavior of the system*. On the theoretical side, uniqueness for the nth-order linear equation underlies the entire theory of the Wronskian and linear independence.

Here is an example pertaining to existence. If a function ϕ has a bounded derivative on a finite interval (a, b), it is easy to show that the endpoint limits

$\phi(a+)$ and $\phi(b-)$ both exist. This allows us to extend a local to a global solution without the familiar complications entailed by an oscillatory approach to the boundary. Again the proof is easy, the ramifications significant.

Also related to existence is the fact that convergence of power-series solutions can be established by an easy mathematical induction; the method of majorants is not needed. For boundary-value problems we now have Chernoff's novel approach to convergence of Fourier series. When extended to piecewise smooth functions, it opens up a subject that until then was almost forbidden territory.

If the theory of exact differentials $\omega = Pdx + Qdy$ is developed by use of definite integrals, as it should be, one can readily see why the main result might be expected to fail for ring-shaped regions. Thus the way is prepared for simple connectivity and line integrals as encountered in more advanced courses.

The familiar theorem asserting linear independence of $\{t^j e^{st}\}$ is both important and easy to prove. It yields a basis for solutions of the general linear equation with constant coefficients, it plays an essential role in the theory of asymptotic stability, and it is a logical prerequisite for the method of annihilators. From a broader perspective, *elementary differential equations* is perhaps the first place where students learn to use complex numbers with skill and confidence.

These remarks may serve to emphasize the fertile interaction between easy proofs, uncomplicated ideas, and significant results, that is offered by the study of differential equations. The interaction provides an ideal introduction to more austere courses of modern algebra and advanced analysis, and it has the potential to make a significant contribution to mathematics education.

On the world scene, this potential is actually realized. For example Wolfgang Walter's marvelously brief and elegant text proves virtually everything, even the spectral theorem for compact self-adjoint operators! Yet it has been, for a long time, the most popular undergraduate text on differential equations in Germany.

Current trends as set by textbooks

Returning to reality, let us look at some of the most popular texts currently used in this country. Typically, key formulas and theorems are highlighted by colored print, or displayed in boxes, so students can do the homework without reading the book. The text itself is characterized by the omission of proofs "without apology," as is sometimes said, or by the boast "no frills" which really means "no proofs."

For example, what about uniqueness? One strategy is to discuss the second-order linear equation with positive constant coefficients, by a proof that does not generalize, and let it go at that. Another strategy is to discuss Picard iteration in the one-dimensional case, typically through a mixture of text and problems in which uniform convergence is barely mentioned. It is then said, without further details, that the alleged "proof" extends to vectors and thence to higher-order equations. But closer examination indicates that uniqueness has not really been proved even for the second-order linear equation. Therefore the consequences of uniqueness have not been proved either.

Global existence and convergence of series solutions are omitted from nearly all popular books. However, one much-used text deals with the convergence of Fourier series in the following enterprising fashion: By partial integration, it is shown that the coefficients of a C^2 periodic function are of order $1/n^2$, hence the series is uniformly convergent. This completes the proof. The question whether the series converges to the function that generated it is not addressed at all.

More surprising is the fact that the theory of exact differentials is commonly based on indefinite integrals, the equation

$$\frac{\partial}{\partial y}\int f(x, y)\,dx = \int f_y(x, y)\,dx$$

being used as a disguised form of $F_{yx} = F_{xy}$. The trouble with this approach is that it is local. If students are asked why the proof fails for a ring-shaped region, they cannot answer, because the role of simple connectivity is effectively concealed. Yet a development by definite integrals can be found in books written thirty or more years ago. Retrogression is not progress.

In most popular books linear independence of $\{t^j e^{st}\}$ is established only for two functions, or for the case $j = 0$, or sometimes in an additional case in which all exponents s are equal. But the applications mentioned above require the general theorem; special cases are not enough. If the theorem is unproved, its consequences are also unproved.

Confident and skillful use of complex numbers is seldom seen in popular texts, with the result that our math students are kept far behind those in physics and engineering. In fact at least one well-known book casts doubt on the familiar formula for differentiating $\exp(a + ib)t$ and suggests that results obtained with complex numbers must be checked by use of the corresponding real form! Yet justification of the key formula is trivial; merely differentiate the defining relations

$$e^{ibt} = \cos bt + i\sin bt, \qquad e^{(a+ib)t} = e^{at}e^{ibt}.$$

To paraphrase Madame Roland: O rigor! what things are done in thy name!

Since complex numbers are avoided as far as possible, students typically are asked to substitute an expression like

$$t^2 e^{at}(A\cos bt + B\sin bt)$$

into a high-order differential equation to get the constants A and B. Even more typically, they are asked to write the form of a trial solution and let it go at that. Anyone faced with a genuine problem in mechanical or electrical engineering may be forgiven for doubting the utility of this long-outmoded procedure.

Zwillinger's book

With these remarks as background, let us look at the book under review. Since proofs are systematically omitted, it might be thought that this reviewer would object. But, not so. Zwillinger's aims are entirely different from those set forth above. His book is more like Pierce's *Table of integrals* than like a conventional text. The emphasis is quite properly not on proofs, but on correct statements, and there is no attempt to give an appearance of rigor without the reality.

Within this self-imposed limitation, what a wealth of interesting results in small compass! Zwillinger makes consistent use of the format:

applicable to, yield, idea, procedure.

The first item describes the kind of problems the method pertains to. If your problem fails to qualify, read no further. The second says what the method will do. The basic idea is explained next, and finally comes a description of the procedural details. A most happy feature of the exposition is that every method is illustrated by at least one well-chosen example. For example, bifurcation is illustrated by a bead on a ring that rotates about a vertical diameter. Equating forces makes it clear why there will be two equilibrium positions when the rotation is rapid, only one when it is slow. The differential equation then allows us to decide about stability. Equally impressive is an illustration of interval analysis in connection with Picard's method. One example like these is worth many words.

The book must have involved an immense amount of labor, and as a consequence, it has far too much content to be summarized here. For example, §33 entitled "Look Up Technique" contains about 250 equations with names and references, and the bibliography for this section alone has 112 references. Here is a list of the main headings in the Table of Contents. On the average, each is accompanied by about thirty subheads:

Definitions and concepts
Transformations
Exact analytical methods, ODEs
Exact analytical methods, PDEs
Approximate analytical methods
Numerical methods: concepts
Numerical methods for ODEs
Numerical methods for PDEs

An important feature is that the book is typeset by the author himself, using $\TeX$. No doubt this helps to make the relatively low price possible, and it eliminates the usual error-generating transfer from manuscript to printed page. (I found only two misprints, both unimportant.) The attractiveness of the typography suggests that the author is a skilled compositor as well as a skilled expositor.

It is to be expected, of course, that every reader can come up with favorite topics that are excluded. I should have liked to see LaSalle's improvement of Liapunov's method, a better development of the matrix Riccati equation and its bearing on scattering and transfer, a more up-to-date account of maximum principles, the concept of the graph of a matrix and its relation to Lotka-Volterra systems, and further details about monotone operators, to mention five examples. Also virtually no methods are discussed in depth. But if more topics had been included, and each had been fully developed, the book would have been over 7000 pages long, not less than 700. The author's decision to omit proofs is dictated by this same requirement of reasonable size.

One aspect of the exposition to which I take exception is Zwillinger's preference for expository accounts and (in my view undistinguished) elementary texts over original sources. Too often the treatment in these references is decidedly superficial, and they also distort the historical record. For example, the main sources for the line method in parabolic differential equations are all omitted.

I think the book could be a good supplement to a course on differential equations, but, as is clear from the early part of this review, I would be aghast if it were the main text. That would be like using a book of cooking recipes as the main text for organic chemistry. If we mathematicians abandon the goal of logical development, who will replace us?

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Retarded dynamical systems: stability and characteristic functions, by G. Stépán. Longman Scientific & Technical, Pitman Research Notes in Mathematics Series, no. 210, 1989, 151 pp; $32.00. ISBN 0-582-03932-0

In engineering, biology, and physics, one often encounters dynamical systems that may be described a systems with memory, or hereditary systems, or systems with delayed feedback or time lag. The mathematical formulation and basic theory of the differential equations that describe such systems may be said to have begun with the work of A. D. Myshkis [9] and since that time there has been a growing body of mathematical research on many aspects of theory and applications, and the development of a general theory for what are now called functional differential equations (FDEs). The present book is devoted to an aspect of great practical importance, the stability theory for these equations. As will be explained below, the local stability theory depends on analysis of the location of zeros of associated "characteristic functions." A variety of methods have been proposed for treating the stability problem. Among these is the method of Pontryagin, described in [10] and in the book of Bellman and Cooke [1], but it is complicated for equations with more than one delay. Another is the D-subdivision or D-partition method, in which the space of the parameters of the equation is divided by hypersurfaces, the points of which correspond to quasipolynomials having at least one zero on the imaginary axis. This method, and others such as the tau-decomposition method and Nyquist criterion are described thoroughly in the books of El'sgol'ts and Norkin [3], MacDonald [8], and Kolmanovskii and Nosov [6]. Besides these, Liapunov functional techniques (see Hale [4], Yoshizawa [12]) are sometimes useful for either linear or nonlinear problems. Some general results for one delay equations are given in Cooke and van den Driessche [2]. (G. Boese has pointed out that hypothesis (iv) in Theorem 1 must be strengthened in the general case.)

Stépán comments, with considerable justification, that "none of these methods can be used generally for functional differential equations." For instance, the widely-used D-subdivision method depends heavily upon the knowledge of the hypersurfaces, which is generally difficult to find. His book is devoted to this problem, and consists of two main parts. The first part is the explanation of his own method, which he calls the "direct stability investigation," and which is presented with full proofs in Chapter 2. The second part is the construction of so-called stability charts, carried out for many equations with the aid of his direct method in Chapters 3 and 4. In sum, these provide very useful and quite broadly applicable tools for handling the stability problem.